

UNBIASED FREQUENCY ESTIMATION OF NARROWBAND SIGNALS USING PROCRUSTES TYPE SUBSPACE ROTATION

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ABSTRACT

We consider an improved least square algorithm based on subspace rotations using Procrustes approximation. Robust estimation of the poles of a narrowband signal is achieved by reducing the bias of the estimates. The key idea is to consider the signal as a vector in multi-dimensional space, and to separate the "pure signal" and the noise into two mutually orthogonal components lying in different subspaces. Though the quality of the estimated signal is not much improved as compared to the conventional least square based subspace approaches, the advantage of using the proposed algorithm is the asymptotically unbiased nature of the estimates.

1. INTRODUCTION

The objective of robust estimation is to improve noisy signals and also to obtain robust estimates of the signal parameters. An important application is the enhancement of narrowband signals degraded by broadband noise, as in the case of hands-free mobile telephony where speech communication is affected by the presence of acoustic noise. This effect is particularly serious when linear predictive coding (LPC) is used for digital representation of speech signals at low bit rates as in digital mobile telephony. Therefore, there is a great need for using robust noise reduction approaches leading to a better set of LPC coefficients for the synthesis stage.

Several signal estimation approaches have been proposed, and two categories of algorithms dominate in the current literature. One of them is based on spectral decomposition [1] and the other is based on structured approaches for denoising [2,3,4]. The latter method uses least squares (LS), total least squares (TLS) or partial least squares (PLS) approaches for estimating a clean signal. Though LS based approaches are computationally more viable, they suffer from biased parameter estimates due to an improper noise structure in the model [5]. In the present paper, we use a modified LS approach based on

the signal subspace paradigm, which overcomes the above limitation.

Signal subspace methods have been used frequently in connection with, for example, frequency estimation problems [3,4]. In all these approaches, the data is first arranged in a Hankel/Toeplitz form. A LS estimate of the signal-only data matrix is then obtained, which preserves the structured form of the matrix. The key idea is to consider the signal as a vector in multi-dimensional space (of dimension N), and to separate the "pure signal" (of dimension p) and the noise into two mutually orthogonal components lying in different subspaces. It is well known that with degradation of the SNR, the predictability of the signal deteriorates, with the result that the dimension p to be chosen becomes a large fraction of N . A further consequence of low SNR is that the signal subspace eigenvectors, i.e. those corresponding to the large and well separated (in magnitude) eigenvalues and which are less perturbed from their noiseless direction become less in number as compared to those eigenvectors that change direction depending on the noise perturbation. In the proposed method, we show that the LS estimation performed on a matrix constructed by projecting the columns of the reduced rank approximant of the data matrix onto an orthonormal set of basis vectors results in a reduced spread of the eigenvalues in the signal subspace. This leads to an improved accuracy of the parameter estimates involving large subspace dimensions.

2. LS PROJECTION ALGORITHM USING PROCRUSTES ROTATIONS

The noisy signal z is represented by the additive noise model

$$z(n) = x(n) + w(n), \quad n = 0, 1, \dots, N-1 \quad (1)$$

where z represents the noisy signal, x represents the clean signal and w represents i.i.d Gaussian noise. We define the data matrices $X^{(1)}$ and $X^{(2)}$ to be:

$$\begin{aligned} \mathbf{X}^{(1)} &= (\mathbf{x}_p, \mathbf{x}_{p-1}, \dots, \mathbf{x}_0) \\ &= \mathbf{X} \end{aligned} \quad (2)$$

and

$$\mathbf{X}^{(2)} = (\mathbf{x}_{p+1}, \mathbf{x}_p, \dots, \mathbf{x}_1) \quad (3)$$

In the noise-free case, the matrices $\mathbf{X}^{(1)}$ and $\mathbf{X}^{(2)}$ have rank d and span the same column space and row space. Under noisy conditions, the matrices $\mathbf{Z}^{(1)}$ and $\mathbf{Z}^{(2)}$ are of full rank $p+1$ and they no longer span the same column space and row space. A key step that is followed in our method is the search for a common d -dimensional subspace obtained as the intersection of the subspaces denoted by the matrix pair $\mathbf{Z}^{(1)}$ and $\mathbf{Z}^{(2)}$. The Procrustes approximation of the intersecting space is then carried out with a view to construct an orthonormal basis onto which the measured data vectors are projected to obtain a rank- d approximation of the signal vectors in the signal domain. The Hankel matrix representing the signal subspace is reconstructed by projecting the column space of the reduced rank- d LS approximation of the noisy matrix onto an orthonormal set of basis vectors representing the common d -dimensional subspace constructed via the Procrustes approximation. A physical interpretation of this result may be attributed to the statistical orthogonality existing between the signal and noise components, which accounts for the reduced amount of noise components being projected onto the cleaned up version of the intersecting subspace obtained via the Procrustes approximation. The desired orthonormal bases are obtained via the QR factorizations of the respective data matrices.

Let $\mathcal{E}$ and $\mathcal{W}$ denote the spaces spanned by the column space of $\mathbf{Z}^{(1)}$ and $\mathbf{Z}^{(2)}$ in R^{p+1} . If the columns of $\mathbf{Q}_\mathcal{E}$ and $\mathbf{Q}_\mathcal{W}$ define orthonormal bases for $\mathcal{E}$ and $\mathcal{W}$ respectively, then

$$\begin{aligned} &\max_{e \in \mathcal{E}} |e|_2 = 1 \quad \max_{w \in \mathcal{W}} |w|_2 = 1 \quad (\mathbf{E}^T \mathbf{W}) \\ &= \max_{t \in R^p} |t|_2 = 1 \quad (\mathbf{S}^T (\mathbf{Q}_\mathcal{E}^T \mathbf{Q}_\mathcal{W}) \mathbf{T}) \end{aligned} \quad (4)$$

Using the minimax characterization of the singular values,

$$\mathbf{S}^T (\mathbf{Q}_\mathcal{E}^T \mathbf{Q}_\mathcal{W}) \mathbf{T} = \text{diag}(\zeta_1, \dots, \zeta_p) \quad (5)$$

is the SVD of $\mathbf{Q}_\mathcal{E}^T \mathbf{Q}_\mathcal{W}$. The principal angles

$\{\theta_1, \dots, \theta_{p+1}\} \in [0, \frac{\pi}{2}]$ between $\mathcal{E}$ and $\mathcal{W}$ are defined by [6]:

$$\cos(\theta_k) = \zeta_k \quad (6)$$

for $k = 1, \dots, p$. The principal vector matrices $\mathbf{E}$ and $\mathbf{W}$ are defined as:

$$\begin{aligned} \mathbf{E} &= [\mathbf{e}_1, \dots, \mathbf{e}_{p+1}] \\ &= \mathbf{Q}_\mathcal{E} \mathbf{S} \end{aligned} \quad (7)$$

and

$$\begin{aligned} \mathbf{W} &= [\mathbf{w}_1, \dots, \mathbf{w}_{p+1}] \\ &= \mathbf{Q}_\mathcal{W} \mathbf{T} \end{aligned} \quad (8)$$

In the noise-free case, the matrix pair $\mathbf{X}^{(1)}$ and $\mathbf{X}^{(2)}$ span the same d -dimensional space. The QR factorization of the noise-free matrix pair is then given by:

$$[\mathbf{Q}_1 | \mathbf{Q}_1^\perp]^T \mathbf{X}^{(1)} \mathbf{P}_1 = \begin{pmatrix} \mathbf{R}_{11} & \mathbf{R}_{12} \\ \mathbf{O} & \mathbf{O} \end{pmatrix} \quad (9)$$

and

$$[\mathbf{Q}_2 | \mathbf{Q}_2^\perp]^T \mathbf{X}^{(2)} \mathbf{P}_2 = \begin{pmatrix} \mathbf{R}_{21} & \mathbf{R}_{22} \\ \mathbf{O} & \mathbf{O} \end{pmatrix} \quad (10)$$

Since the matrix pair spans the same column space, it can be easily verified that

$$\mathbf{Q}_1^T \mathbf{Q}_2 = \mathbf{Q}_2^T \mathbf{Q}_1 = \mathbf{I}_{d \times d} \quad (11)$$

where $[\mathbf{Q} | \mathbf{Q}^\perp]$ is a unitary square matrix and $\mathbf{P}_1$ and $\mathbf{P}_2$ denote permutation matrices. The d columns of the submatrix $\mathbf{Q}$ represent the orthonormal set of basis vectors for the subspace pair. $\mathbf{R}_{11}$ and $\mathbf{R}_{21}$ represent nonsingular upper triangular matrices of order d . Under perturbed conditions, the matrix pair $\mathbf{Z}^{(1)}$ and $\mathbf{Z}^{(2)}$ have full column rank and hence the desired orthonormal basis vectors can be obtained via the QR factorization of the reduced rank- d approximants of the matrix pair, which yields:

$$[\tilde{\mathbf{Q}}_1 | \hat{\mathbf{Q}}_1^\perp]^T \hat{\mathbf{Z}}^{(1)} \mathbf{P}_1 = \begin{pmatrix} \hat{\mathbf{R}}_{11} & \hat{\mathbf{R}}_{12} \\ \mathbf{O} & \mathbf{O} \end{pmatrix} \quad (12)$$

and

$$[\hat{\mathbf{Q}}_2 | \tilde{\mathbf{Q}}_2^\perp]^T \hat{\mathbf{Z}}^{(2)} \mathbf{P}_2 = \begin{pmatrix} \hat{\mathbf{R}}_{21} & \hat{\mathbf{R}}_{22} \\ \mathbf{O} & \mathbf{O} \end{pmatrix} \quad (13)$$

For the noisy case, the orthonormal basis vectors for the d -dimensional subspace is constructed from the SVD of the matrix $\tilde{\mathbf{Q}}_{1,2} = \tilde{\mathbf{Q}}_1^T \tilde{\mathbf{Q}}_2$, which is no longer unitary (unlike in the noise-free case). The SVD of $\tilde{\mathbf{Q}}_{1,2}$ is expressed as:

$$\tilde{\mathbf{Q}}_{1,2} = \tilde{\mathbf{S}} \tilde{\mathbf{\Gamma}} \tilde{\mathbf{T}} \quad (14)$$

where $\tilde{\mathbf{\Gamma}} = \text{diag}(\zeta_1, \dots, \zeta_d)$. The dimension of the intersecting subspace corresponds to the number of principal singular values, which is close to unity. A rank- d Procrustes approximant is constructed by replacing the d singular values of $\tilde{\mathbf{\Gamma}}$ to unity, denoted by:

$$\hat{\tilde{\mathbf{Q}}}_{1,2} = \hat{\mathbf{S}} \hat{\mathbf{T}}^T \quad (15)$$

where $\tilde{\mathbf{Q}}_{1,2} = \hat{\mathbf{Q}}_1^T \hat{\mathbf{Q}}_2$ is a unitary matrix. The orthonormal basis for the common d -dimensional subspace is then evaluated using:

$$\mathbf{E} = \hat{\mathbf{Q}}_1 \hat{\mathbf{S}} \quad (16)$$

and

$$\mathbf{W} = \hat{\mathbf{Q}}_2 \hat{\mathbf{T}} \quad (17)$$

where $\hat{\mathbf{Q}}_1$ and $\hat{\mathbf{Q}}_2$ denote matrices having orthonormal columns and satisfying the relationship

$$\hat{\mathbf{Q}}_1^T \hat{\mathbf{Q}}_2 = \hat{\mathbf{Q}}_{1,2}. \quad (18)$$

The intersecting space denoted by $R(\mathbf{Z}^{(1)}) \cap R(\mathbf{Z}^{(2)})$ is given by:

$$\begin{aligned} R(\mathbf{Z}^{(1)}) \cap R(\mathbf{Z}^{(2)}) &= \text{span}\{\mathbf{E}\} \\ &= \text{span}\{\mathbf{e}_1, \dots, \mathbf{e}_d\} \\ &= \text{span}\{\mathbf{W}\} \\ &= \text{span}\{\mathbf{w}_1, \dots, \mathbf{w}_d\} \end{aligned} \quad (19)$$

The equality between the two sets of orthonormal vectors results from the observation that if $\zeta_k = 1$, then $\mathbf{e}_k = \mathbf{w}_k$ and leads to the condition

$$\hat{\mathbf{Q}}_1 \hat{\mathbf{S}} = \hat{\mathbf{Q}}_2 \hat{\mathbf{T}}. \quad (20)$$

It is clear from Eqs. (14)-(18) that the matrices $\hat{\mathbf{Q}}_1$ and $\hat{\mathbf{Q}}_2$ can be expressed as:

$$\hat{\mathbf{Q}}_1 = \mathbf{E} \hat{\mathbf{S}}^T \quad (21)$$

$$\hat{\mathbf{Q}}_2 = \mathbf{E} \hat{\mathbf{T}}^T \quad (22)$$

The matrix $\mathbf{E}$ is determined by minimizing $\|\Phi - \mathbf{E} \mathbf{Y}^T\|_F$ where $\Phi = [\hat{\mathbf{Q}}_1, \hat{\mathbf{Q}}_2]$ and $\mathbf{Y}^T = [\hat{\mathbf{S}}^T, \hat{\mathbf{T}}^T]$.

Theorem 1: Let Φ , $\mathbf{E}$ and $\mathbf{Y}$ represent matrices of dimensions $m \times q$, $m \times n$ and $q \times r$ respectively with $m > r$, q . The matrix $\mathbf{E}$ with orthonormal columns that minimizes $\|\Phi - \mathbf{E} \mathbf{Y}^T\|_F$ is such that $\mathbf{E} = \hat{\Pi}^T \Xi$, where Ξ and Π denote the left and right singular vector matrices of $\mathbf{Y}^T \Phi^T$, respectively, and where $\hat{\Pi}$ represents the first r columns of the matrix Π . The objective function to be minimized is equivalent to that of minimizing $\|\Phi - \mathbf{E} \mathbf{Y}^T\|_F$:

$$\begin{aligned} &\|\Phi - \mathbf{E} \mathbf{Y}^T\|_F^2 \\ &= \text{Tr}\{(\Phi - \mathbf{E} \mathbf{Y}^T)(\Phi^T - \mathbf{Y} \mathbf{E}^T)\} \\ &= \text{Tr}\{\Phi \Phi^T\} + \text{Tr}\{\mathbf{E} \mathbf{Y}^T \mathbf{Y} \mathbf{E}^T\} - 2 \text{Tr}\{\mathbf{E} \mathbf{Y}^T \Phi^T\} \end{aligned}$$

$$= \text{Tr}\{\Phi \Phi^T\} + \text{Tr}\{\mathbf{Y}^T \mathbf{Y}\} - 2 \text{Tr}\{\mathbf{E} \mathbf{Y}^T \Phi^T\} \quad (23)$$

From Eq. (23), it is clear that minimizing $\|\Phi - \mathbf{E} \mathbf{Y}^T\|_F$ is equivalent to maximizing $\text{Tr}\{\mathbf{E} \mathbf{Y}^T \Phi^T\}$. Also, let the SVD of $\mathbf{Y}^T \Phi^T$ be expressed as

$$\mathbf{Y}^T \Phi^T = \Xi \Lambda \Pi^T. \quad (24)$$

From Eq. (24),

$$\begin{aligned} \text{Tr}\{\mathbf{E} \mathbf{Y}^T \Phi^T\} &= \text{Tr}\{\mathbf{E} \Xi \Lambda \Pi^T\} \\ &= \text{Tr}\{\Pi^T \mathbf{E} \Xi \Lambda\} \\ &= \text{Tr}\{\Theta \Lambda\} \\ &= \sum_{i=1}^m \sum_{j=1}^r \theta_{ij} \delta_{ji} \\ &= \sum_{i=1}^r \theta_{ii} \delta_{ii} \end{aligned} \quad (25)$$

(since $\delta_{ji} = 0$ for $i \neq j$ and $i > r$)

Since Θ is of rank r and has orthonormal columns, $\|\theta_{ij}\| \leq 1$. Hence, an upper bound for the trace in Eq. (25) is obtained by setting $\theta_{ii} = 1$, so that

$$\text{Tr}\{\mathbf{E} \mathbf{Y}^T \Phi^T\} \leq \sum_{i=1}^r \delta_{ii}. \quad (26)$$

Again, using the fact that Θ has orthonormal columns, the upper bound is obtained for

$$\Theta = \begin{pmatrix} \mathbf{I}_{r \times r} \\ \dots \\ \mathbf{O} \end{pmatrix}. \quad (27)$$

The minimizing value of $\mathbf{E}$ follows from the definition of Θ and is given by:

$$\begin{aligned} \mathbf{E} &= \Pi \begin{pmatrix} \mathbf{I}_{r \times r} \\ \dots \\ \mathbf{O} \end{pmatrix} \Xi^T \\ &= \hat{\Pi} \Xi^T. \end{aligned} \quad (28)$$

With $m = N - p$, $q = 2d$ and $r = d$, the estimation of $\mathbf{E}$ can be performed by extending Theorem 1 to the case where $\Phi = [\hat{\mathbf{Q}}_1, \hat{\mathbf{Q}}_2]$ and $\mathbf{Y}^T = [\hat{\mathbf{S}}^T, \hat{\mathbf{T}}^T]$. The filtered data matrix $\hat{\mathbf{X}}$ is then obtained by projecting the noisy matrix $\hat{\mathbf{Z}}^{(1)}$ onto the column space of $\mathbf{E}$:

$$\begin{aligned} \hat{\mathbf{X}} &= \mathbf{E} \mathbf{E}^T \hat{\mathbf{Z}}^{(1)} \\ &= \hat{\Pi} \hat{\Pi}^T \hat{\mathbf{Z}}^{(1)} \end{aligned} \quad (29)$$

The matrix $\hat{\mathbf{X}}$ is no longer in Hankel form. $\hat{\mathbf{X}}$ is transformed into Hankel form by averaging the elements of the cross diagonal elements [4]. The samples of the noise-free signal $\hat{x}_n$ are then given by the vector formed by lining up the elements of the first column and last row of the Hankel form of $\hat{\mathbf{X}}$ in a contiguous manner.

3. SIMULATIONS

Data was generated according to the ARMA (2,2) model $y(n) - 0.8y(n-1) + 0.65y(n-2) = u(n) + u(n-2)$. The poles are located at $0.4 \pm 0.7j$. The input $u(n)$ was drawn from an i.i.d, zero mean, unity variance, Gaussian-distributed noise. Thirty independent realizations of the noisy data, containing $z(n) = y(n) + w(n)$, were generated, each consisting of 1024 points. The variance of $w(n)$ was adjusted in order to obtain SNRs of 20, 10, and 0 dB. Normalized singular values of the matrices used in our method suggests an AR order of 2. The realizations were performed for subspace dimensions ranging from 2 to 10. Fig. 1(a) shows the angular frequency estimates obtained by using standard LS. Fig. 1(b) shows the plots of the angular frequencies estimated using the proposed approach.

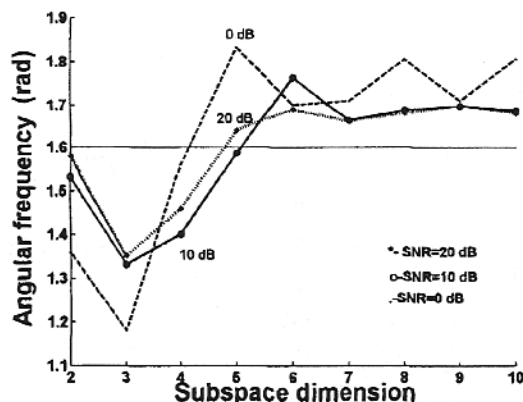


Figure 1(a). Angular frequency estimates obtained with standard LS.

Each figure shows the estimates obtained for SNRs of 20 dB, 10 dB and 0 dB. It is clear that the estimates obtained using the Procrustes approximation of the intersecting subspaces are clearly asymptotically unbiased, while the conventional method displays a clear bias on the estimated poles of the system. In both methods, the variance of the estimates is shown to increase with reduction in SNR.

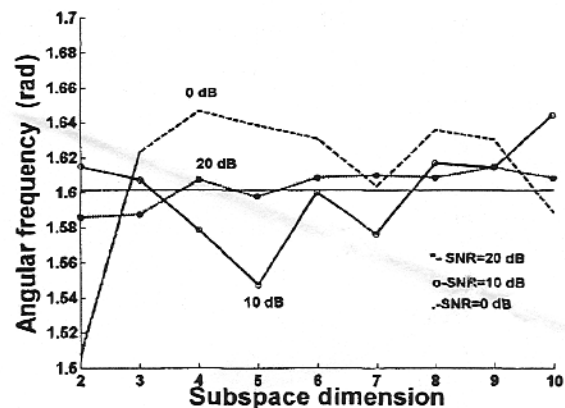


Figure 1(b). Angular frequency estimates obtained with LS projection using Procrustes rotation.

4. REFERENCES

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