# Quantum Chromodynamics

QCD: An Example of a Gauge Theory

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# What is QCD?

1. A gauge theory with a local SU(3) 'color' symmetry;
2. A manifestly gauge-invariant theory defined on a lattice in 3+1 Euclidean space-time
3. An Effective Field Theory of massless Goldstone Bosons (pions, kaons) interacting with a broken chiral symmetry

- Each is a valid definition, with useful and powerful computational consequences
- Each is incomplete, and cannot fully describe the physics of the strong interaction without the other two.

# Dirac Equation

- Consider the Lagrange density of a free dirac particle:

$$\mathcal{L}(x) = \sum_{\mu=0}^{3} \bar{\psi}(x)\left[i\hbar\gamma^{\mu}\frac{\partial}{\partial x^{\mu}} - m\right]\psi(x) \equiv \bar{\psi}(x)\left[i\hbar\gamma^{\mu}\partial_{\mu} - m\right]\psi(x)$$

  - The gamma-matrices are representations of the Clifford Algebra:

$$\left\{\gamma^{\mu},\gamma^{\nu}\right\} = \gamma^{\mu}\gamma^{\nu} + \gamma^{\nu}\gamma^{\mu} = 2\mathbf{I}g^{\mu\nu} \qquad \bar{\psi}(x) = \psi^{\dagger}(x)\gamma_0$$

  - The components of the metric tensor $g^{\mu\nu} = g_{\mu\nu}$ :

    $g^{00}=1=-g^{jj} \quad j=1,2,3; \qquad g^{0j}=0=g^{jk} \quad j\neq k$

  - Upper and Lower components:

    $A^{\mu} = [A^0, \mathbf{A}], \qquad A_{\mu} = g_{\mu\nu} A^{\nu} = [A^0, -\mathbf{A}]$

  - The Action: $S = \int d^4x\mathcal{L}(x)$

  - The classical equations of motion

$$\frac{\delta S}{\delta\bar{\psi}(x)} = 0 \quad \Rightarrow \quad \left[i\hbar\gamma^{\mu}\partial_{\mu} - m\right]\psi(x) = 0$$

# Representations of the Dirac Matrices

- Dirac Representation of gamma-matrices:

$$\gamma^0 = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}, \quad \gamma^j = \begin{bmatrix} 0 & \sigma_j \\ -\sigma_j & 0 \end{bmatrix}$$

$$\left[\sigma_j, \sigma_k\right] = i\varepsilon_{jkl}\sigma_l \quad \Rightarrow \quad \sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

  - Dirac Equation:

$$\left[i\gamma^\mu \partial_\mu - m\right]\psi(x) = 0$$

  - Solutions:

$$\psi(x) = \sqrt{E+m}\begin{bmatrix} \chi(m_z) \\ \dfrac{\sigma \cdot p}{E+m}\chi(m_z) \end{bmatrix} e^{-ix\cdot p/\hbar}, \quad \sigma_z \chi(m_z) = 2m_z \chi(m_z), \quad 2m_z = \pm 1$$

$$\psi^\dagger(x)\psi(x) = 2E, \qquad \bar{\psi}(x)\psi(x) = 2m$$

# Gauge Invariance

- The Lagrange density is invariant under a global phase change: $\psi(x) \rightarrow e^{i\alpha}\psi(x)$
  - Causality suggests that the dynamics should be invariant under local phase changes: $\psi(x) \rightarrow e^{i\Lambda(x)}\psi(x)$
  - The dynamics in Norfolk should not be affected by an instantaneous phase change of the wave-function in Boston.
- Consider the variation of the Action: $S = \int \mathcal{L} d^4x$ as $\psi(x) \rightarrow e^{i\Lambda(x)}\psi(x)$

$$S \rightarrow S' = \int \bar{\psi}(x) e^{-i\Lambda(x)} \left[ i\gamma^{\mu}\partial_{\mu} - m \right] e^{i\Lambda(x)} \psi(x) d^4x$$

$$= \int \bar{\psi}(x) \left\{ i\gamma^{\mu}\partial_{\mu} - \left[ \partial_{\mu}\Lambda(x) \right] - m \right\} \psi(x) d^4x$$

- The Action is invariant if we make the "minimal substitution": $\partial_\mu \rightarrow D_\mu = \partial_\mu - ieA_\mu$ with $A_\mu$ transforming as $A_\mu \rightarrow A_\mu + \partial_\mu \Lambda(x)/e$

# QED Lagrangian

$$\mathcal{L}(x) = \bar{\psi}(x)\left[\gamma^{\mu}\left(i\hbar\partial_{\mu} - eA_{\mu}(x)\right) - m\right]\psi(x) + \frac{1}{4}F_{\mu\nu}(x)F^{\mu\nu}(x)$$

$$F_{\mu\nu}(x) = \partial_{\mu}A_{\nu}(x) - \partial_{\nu}A_{\mu}(x)$$

$$S = \int \mathcal{L}(x)d^4x$$

- Classical Equations of motion:

$$\frac{\delta S}{\delta\bar{\psi}(x)} = 0 \quad \Rightarrow \quad \gamma^{\mu}\left(i\hbar\partial_{\mu} - eA_{\mu}(x)\right)\psi(x) = m\psi(x)$$

$$\frac{\delta S}{\delta A_{\mu}(x)} = 0 \quad \Rightarrow \quad \partial_{\nu}F^{\mu\nu}(x) = e\bar{\psi}(x)\gamma^{\mu}\psi(x) = j^{\mu}(x)$$

- Gauge Invariance → $F^{\mu\nu}$ antisymmetric→Conserved current $\partial_{\mu}j^{\mu}=0$
- Maxwell Equations (Gaussian units):

$$-\partial_{\nu}F^{0\nu}(x) = \vec{\nabla}\cdot\left[-\partial^0\vec{A} + \vec{\nabla}A^0\right] = \vec{\nabla}\cdot E = j^0(x) = \rho(x)$$

$$-\vec{e}_i\,\partial_j F^{ij}(x) = \vec{\nabla}\times\left(\vec{\nabla}\times\vec{A}\right) = \vec{\nabla}\times\vec{B} = \vec{j}(x)$$

# The QED Lagrangian, Now What?

- Path Integral Formalism: Action with sources:

$$Z[J] = \int D\psi D\bar{\psi} DA e^{iS[\psi,\bar{\psi},A;J]/\hbar}$$

$$S[\psi,\bar{\psi},A;J] = \int d^4x \left[\mathcal{L}^{(0)}(x) + \mathcal{L}^{(I)}(x) + \mathcal{L}^{(S)}(x)\right]$$

$$\mathcal{L}^{(0)}(x) = \bar{\psi}(x)\left[i\hbar\gamma\cdot\partial - m\right]\psi(x) + \frac{1}{4}F_{\mu\nu}(x)F^{\mu\nu}(x)$$

$$\mathcal{L}^{(I)}(x) = -e\bar{\psi}(x)\gamma\cdot A(x)\psi(x)$$

$$\mathcal{L}^{(S)}(x) = \bar{\psi}(x)J_{\mu}(x)\psi(x) + \ldots$$

- Expand *Z[J]* in powers of the interaction and source terms
  - Perturbation expansion in powers of $\alpha$.
  - Feynman Diagrams
  - Challenge to restore Gauge Invariance.

# Hamiltonian Formalism

- Momentum field conjugate to particle field

$$\pi(x) = \frac{\delta S}{\delta \dot{\psi}(x)} = i\psi^\dagger(x)$$

$$\vec{\Pi}(x) = \frac{\delta S}{-\delta \dot{\vec{A}}(x)} = \vec{E}(x)$$

- Pick a rest-frame
  - Equal Time Quantization (Dirac Fields anti-commute):

$$\left\{\psi_\alpha(\vec{x},t), \pi_\beta(\vec{y},t)\right\} = i\delta_{\alpha,\beta}\delta^{(3)}(\vec{x}-\vec{y}) \quad \Rightarrow \quad \left\{\psi_\alpha(\vec{x},t), \psi_\beta^\dagger(\vec{y},t)\right\} = i\delta_{\alpha,\beta}\delta^{(3)}(\vec{x}-\vec{y})$$

- Hamiltonian Density:

$$\mathcal{H}(x) = \pi(x)\dot{\psi}(x) - \dot{\vec{A}}(x)\cdot\vec{E}(x) - \mathcal{L}(x)$$

$$H = \int d^3\vec{x}\,\mathcal{H}(x)$$

# Dirac's Forms of Hamiltonian Dynamics

- Instant Form
  - Quantize at equal time:
    Similar to non-relativistic Quantum Mechanics.
- Front Form
  - Quantize "on the Light-Cone":
    At equal light front times $x^+ = (ct+z)$
    Hamiltonian = $P^- = (E - P_z)$
- Point Form
  - Quantize at equal proper time $\tau$:
    $\tau^2 = x_\mu x^\mu$

# Abelian vs. Non-Abelian

- The transformation $\psi(x) \rightarrow e^{i\Lambda(x)}\psi(x)$ belongs to the group U(1) of unitary transformations of a complex function. The group is commutative (Abelian).
- SU(2) is the special unitary group defined by all possible unitary transformations,simply-connected to the identity, of the space of 2-component complex vectors:

$$U \in SU(2)$$

$$x = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}, \quad x' = Ux$$

$$x'^{\dagger} x = x^{\dagger} x \Leftrightarrow U^{\dagger} = U^{-1}$$

$$U = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

$$Det[U] = ad - bc = 1,$$

$$UU^{\dagger} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow (aa^* + bb^*) = 1, \quad (ac^* + bd^*) = 0 \ldots$$

# Yang-Mills Theory: A Strong Interaction Failure

- The proton and neutron make up a flavor $SU(2)$ doublet. The pions make up a flavor $SU(2)$ triplet.
- Following the success of QED, Yang and Mills proposed a non-abelian gauge theory of pions coupled to nucleons:
  - $\psi(x,t_z)$ = 4-component dirac spinor $\otimes$ 2-component isospinor:
    - $t_z=+1/2$ = proton spinor, $t_z=-1/2$ neutron
  - $\boldsymbol{\tau} \cdot \mathbf{A}^\mu(x) = \Sigma_a \, \tau_a \, A^{a\mu}(x)$ = pion field = Lorentz vector $\otimes$ $\mathfrak{su}(2)$ isospin matrix
    - $[\tau_a, \tau_b] = i\varepsilon_{abc}\, \tau_c \quad a,b,c = 1,2,3,$
- The $\tau_a$ are $\mathfrak{su}(2)$ isospin matrices (Pauli Matrices):

$$\tau_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \tau_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \tau_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

# Yang-Mills Lagrangian

$$\mathcal{L}_{YM}(x) = \bar{\psi}_N(x)\left[i\gamma^{\mu}\mathbf{D}_{\mu}(x) - m\right]\psi_N(x) + \frac{1}{4}\mathrm{Tr}\left[\mathbf{G}_{\mu\nu}(x)\mathbf{G}^{\mu\nu}(x)\right]$$

$$\mathbf{G}_{\mu\nu}(x) = \partial_{\mu}\mathbf{A}_{\nu}(x) - \partial_{\nu}\mathbf{A}_{\mu}(x) - ig\left[\mathbf{A}_{\mu},\mathbf{A}_{\nu}\right]$$

$$\mathbf{D}_{\mu}(x) = \partial_{\mu} - ig\left[\frac{\boldsymbol{\tau}}{2}\cdot\mathbf{A}_{\mu}(x)\right]$$

- The matrix trace is over the isospin variables
- The "Covariant Derivative" D is an isospin matrix
- Gauge Transformation:

$$\psi(x) \to e^{i\tau_a\theta_a(x)/2}\psi(x) = U(x)\psi(x)$$

$$\boldsymbol{\tau}\cdot\mathbf{A}_{\mu}(x) \to U^{\dagger}(x)\boldsymbol{\tau}\cdot\mathbf{A}_{\mu}U(x) - \frac{1}{g}\boldsymbol{\tau}\cdot\left[\partial_{\mu}\boldsymbol{\theta}(x)\right]$$

$$\mathbf{D}_{\mu}(x)\psi(x) \to U(\boldsymbol{\theta}(x))\mathbf{D}_{\mu}(x)\psi(x)$$

$$\boldsymbol{\tau}\cdot\mathbf{G}_{\mu\nu}(x) \to U(\boldsymbol{\theta}(x))\left[\boldsymbol{\tau}\cdot\mathbf{G}_{\mu\nu}(x)\right]U^{\dagger}(\boldsymbol{\theta}(x))$$

$$\mathrm{Tr}\left[\mathbf{G}_{\mu\nu}(x)\mathbf{G}^{\mu\nu}(x)\right] \to \mathrm{Tr}\left[\mathbf{G}_{\mu\nu}(x)\mathbf{G}^{\mu\nu}(x)\right] \ \text{(invariant)}$$

# From Yang-Mills to ElectroWeak & QCD

- The Lagrangian → Perturbation expansion (Feynman Diagrams) in powers of the $\pi N$ coupling constant $g_{\pi N}$.
  - This involves loop integrals, which diverge.
  - The cure is renormalization, but this only works for a massless gauge boson (the pion, in this case). But pion mass ≠ 0.
- In the 1950's, it seemed that Gauge Theories were a failure at describing nuclear forces.
- 1960s and 1970s revival:
  - Higgs, Englert et al. showed how to give Gauge Bosons mass
    - Weinberg, Glashow, Salam, Electro-Weak Unification: $W^{\pm}$,$Z^0$, H bosons
  - QCD: SU(3) color symmetry, and a hidden massless boson (gluon)

# Massive Bosons

- Gauge symmetry requires massless bosons.
  - Other than the photon, we do not observe massless bosons
- Either:
  - Bosons ($W^{\pm}$,Z) acquire mass via the Higgs-Englert mechanism in the Electro-Weak unification
  - Bosons in QCD (gluons) remain massless, but are hidden by confinement

# The Group SU(3)

- The group defined by Unitary 3x3 matrices: $UU^\dagger = I$
  - $U = e^{i\omega\bullet\lambda}$,
  - $\lambda$ is a vector of $3\times3$ matrices. $\omega$ is a vector of real numbers
  - Infinitessimal element: $1=UU^\dagger \approx (I+i\omega\bullet\lambda)(I-i\omega\bullet\lambda^\dagger) \approx I+i\omega\bullet(\lambda-\lambda^\dagger)$
    The $\lambda$ are hermitian.
  - Det[U] = 1 --> The $\lambda$ are traceless
- Gell-Mann basis:

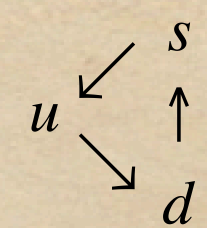
$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \quad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

# Flavor vs. Color SU(3)

- The pattern of hadron masses gave rise to the quark-model of hadrons and baryons, with a flavor SU(3) symmetry.
  - Quarks have flavor u,d,s.
  - Other than a small mass splitting, the dynamics of hadrons are completely invariant under interchange of flavor.
  - Quark spinor = Dirac Spinor $\otimes$ Flavor vector
  - The quark Lagrangian has a flavor SU(3) symmetry (except the mass terms)




- The Quark-model has fully symmetric states: $\Delta^{++} = |uuu\rangle$, $\Omega^{--} = |sss\rangle$...
- To preserve the anti-symmetry of the quark wavefunction, SU(3) color symmetry was introduced.

# From the Quark-Model to the QCD Lagrangian

- The quark model consists of states that are symmetric under combined interchange of flavor and spin.
- Anti-symmetry requires making the quark-fields 3-vectors in a color space. This introduces a global $SU(3)$ symmetry
- Promote this symmetry to a local gauge symmetry, and voilá, we have a theory of interacting quarks.

# QCD Lagrangian

$$\mathcal{L}_{QCD}(x) = \sum_{f=u,d,s,c\ldots} \bar{\psi}_f(x)\left[\gamma^\mu \mathbf{D}_\mu(x) - m_f\right]\psi_f(x) + \frac{1}{4}\mathrm{Tr}\left[\mathbf{G}_{\mu\nu}(x)\mathbf{G}^{\mu\nu}(x)\right]$$

$$\mathbf{G}_{\mu\nu}(x) = \partial_\mu \mathbf{A}_\nu(x) - \partial_\nu \mathbf{A}_\mu(x) - ig\left[\mathbf{A}_\mu, \mathbf{A}_\nu\right]$$

$$\mathbf{A}_\mu(x) = \lambda_a A_\mu^{(a)}(x)$$

$$\mathbf{D}_\mu(x) = \partial_\mu - ig\left[\mathbf{A}_\mu(x)\right]$$

- What can we do with it?
- Asymptotic Freedom allows for perturbation expansion for high momentum transfer processes.
    - $e^+e^-$ → hadrons
    - Deep Inelastic Scattering

# The su(3) Lie-Algebra

- The **λ**-matrices are the generators of SU(3).
    - Lie Algebra: Anti-symmetric "multiplication", which is linear over addition and scaler multiplication
- They can also be defined by their commutation relations:

$$\left[\lambda_a, \lambda_b\right] = \lambda_a \lambda_b - \lambda_b \lambda_a = i f_{a,b}^{c} \lambda_c$$

- The structure constants are antisymmetric under interchange of any two indices, with:

$$f_{1,2}^{3} = 1$$

$$f_{1,4}^{7} = f_{1,6}^{5} = f_{2,4}^{6} = f_{2,5}^{7} = f_{3,4}^{5} = f_{3,7}^{6} = \frac{1}{2}$$

$$f_{4,5}^{8} = f_{6,7}^{8} = \frac{\sqrt{3}}{2}$$

# QCD: Beyond the Lagrangian

- How to calculate low energy phenomena?
  - Hadron Masses
  - NN potential...
- Reformulate the Action:
  - Lattice QCD
    - Quarks on discrete lattice sites
    - Gluon field ***G*** on the links between adjacent sites
      - Gluon Potential *A* disappears from the theory
  - Chiral Perturbation Theory
    - Pions, kaons are manifestation of spontaneous breaking of the chiral symmetry $(1 \pm \gamma_5)\psi$

# Chiral Perturbation Theory

- A Chiral Perturbation Theory Primer arXiv:hep-ph/0505265

### 3.4 The Lowest-Order Effective Lagrangian

Our goal is the construction of the most general theory describing the dynamics of the Goldstone bosons associated with the spontaneous symmetry breakdown in QCD. In the chiral limit, we want the effective Lagrangian to be invariant under $\mathrm{SU}(3)_L \times \mathrm{SU}(3)_R \times \mathrm{U}(1)_V$. It should contain exactly eight pseudoscalar degrees of freedom transforming as an octet under the subgroup $H = \mathrm{SU}(3)_V$. Moreover, taking account of spontaneous symmetry breaking, the ground state should only be invariant under $\mathrm{SU}(3)_V \times \mathrm{U}(1)_V$.

Following the discussion of Section 3.3.2 we collect the dynamical variables in the SU(3) matrix $U(x)$,

$$
\begin{aligned}
U(x) &= \exp\left(i\frac{\phi(x)}{F_0}\right), \\
\phi(x) &= \sum_{a=1}^{8}\lambda_a\phi_a(x) \equiv \begin{pmatrix} \pi^0 + \frac{1}{\sqrt{3}}\eta & \sqrt{2}\pi^+ & \sqrt{2}K^+ \\ \sqrt{2}\pi^- & -\pi^0 + \frac{1}{\sqrt{3}}\eta & \sqrt{2}K^0 \\ \sqrt{2}K^- & \sqrt{2}\bar{K}^0 & -\frac{2}{\sqrt{3}}\eta \end{pmatrix}. \quad (3.28)
\end{aligned}
$$

The most general, chirally invariant, effective Lagrangian density with the minimal number of derivatives reads

$$
\mathcal{L}_{\text{eff}} = \frac{F_0^2}{4}\mathrm{Tr}\left(\partial_\mu U \partial^\mu U^\dagger\right), \quad (3.29)
$$

+ an infinite series of terms with higher derivatives.

# The Quark Model

- The mesons and baryons form degenerate multiplets of a flavor SU(3) symmetry: