

Tessellation & Interpolation

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This note discusses linear interpolation of values on a tessellated mesh of N -simplices

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I. SIMPLEX TESSELLATION IN N -DIMENSIONS

Given an arbitrary collection of n points in N -dimensional space, there is a unique tessellation of N -simplices. An N -simplex is a collection of $N+1$ vertex points $\{\mathbf{X}(i)|i = 0, \dots, N\}$ such that for an arbitrary ordering of the points, the vector set $\{\mathbf{X}(i) - \mathbf{X}(0)|i = 1, \dots, N\}$ form a linearly independent (not necessarily orthogonal) basis for the original N -dimensional space. A 2-simplex is a triangle, a 3-simplex is a tetrahedron, *etcetera*. The basis requirement ensures that, for example, a 3-simplex does not collapse to a triangle.

An N -simplex defines a unique N -sphere, whose surface passes through the $N+1$ vertices. The corresponding open N -ball is the set of all points inside the N -sphere, but not including the sphere. The unique tessellation from the n points is defined by the requirement that **no** vertex is contained in any open N -ball defined by the tessellation.

A. Femtography

The vertices of a tessellation can be constructed from any useful metric. In our current Virginia Center for Nuclear Femtography (CNF) project, we globally minimize the functional differences of the Compton Form Factors between any two adjacent vertices of the simplices.

The particular variables we are interested in are denoted

$$\left[\ln(\xi), t, \ln \left(\frac{Q^2}{1 \text{ GeV}^2} \right) \right] \quad (1)$$

The $\xi \in [0, 1]$ variable describes the momentum of the observed quark. Since quarks are waves, roughly speaking the wavelength of the quark is $\hbar/(Mc\xi) \approx (0.2 \text{ fm})/\xi$. With $1 \text{ fm} = 10^{-15} \text{ meter} = 1 \text{ femto-meter}$. 1 fm is also commonly referred to as “one Fermi”, in honor of Enrico Fermi, a prominent Italian-American physicist of the 20th century and Nobel Laureate. Curiously enough, for $\xi < 0.1$ the quark wavelength is larger than the size of the proton: ultra-low momentum quarks “leak-out”. The variable $t < 0$ is the net momentum transfer squared delivered to the proton in a scattering process. The 2-dimensional impact parameter $\mathbf{b}$ is Fourier-conjugate to $\sqrt{-t}$. Finally, Q^2 is the resolution scale of experiment probing the quark distributions in the proton: specifically: $\hbar c/\sqrt{Q^2}$ is the spatial resolution scale.

In our Femtography space we can define unit vectors $\mathbf{e}(1)$, $\mathbf{e}(2)$, $\mathbf{e}(3)$ as unit vectors in the $\ln(xi)$, t , and $\ln(Q^2)$ “directions”. These unit vectors are orthonormal:

$$\mathbf{e}(i) \cdot \mathbf{e}(j) = \delta_{i,j} = \begin{cases} 1 & \text{for } i = j \\ 0 & \text{for } i \neq j \end{cases} \quad (2)$$

II. LINEAR INTERPOLATION IN AN N -SIMPLEX

A. Coordinates

Let the points $\{\mathbf{X}(i)|i = 0, \dots, N\}$ define an N -simplex S_N , with functional values on the vertices:

$$f_i = f[\mathbf{X}(i)], \quad i = 0, 1, \dots, N. \quad (3)$$

With an arbitrary choice of $\mathbf{X}(0)$, we define a basis set for the simplex as follows:

$$\Delta\mathbf{X}(i) \equiv \mathbf{X}(i) - \mathbf{X}(0), \quad i = 1, 2, \dots, N. \quad (4)$$

In terms of the Cartesian basis $\{\mathbf{e}(1), \mathbf{e}(2), \dots, \mathbf{e}(N)\}$ we have the coordinate transformation Jacobian

$$\begin{aligned} J(i, j) &= \Delta\mathbf{X}(i) \cdot \mathbf{e}(j), \quad (i, j) \in \{1, 2, \dots, N\}, \quad \text{such that} \\ \Delta\mathbf{X}(i) &= \sum_j J(i, j) \mathbf{e}(j) \end{aligned} \quad (5)$$

There is a one-to-one correspondence between the set of points $\mathbf{x} \in S_N$ and the values

$$\left\{ 0 \leq \alpha_i \leq 1 | i = 1 \dots N, \sum \alpha_i \leq 1 \right\}. \quad (6)$$

This correspondence is expressed by

$$\begin{aligned} \mathbf{x} &= \mathbf{X}(0) + \Delta\mathbf{x} \\ \Delta\mathbf{x} &= \sum_{i=1}^N \alpha_i \frac{\mathbf{X}(i) - \mathbf{X}(0)}{|\mathbf{X}(i) - \mathbf{X}(0)|} \quad \forall \mathbf{x} \in S_N \\ &= \sum_{i,j=1}^N \alpha_i \frac{J(i, j) \mathbf{e}_j}{|\Delta\mathbf{X}(i)|} \end{aligned} \quad (7)$$

Our basis vectors are complete and non-degenerate, therefore the matrix $J(i, j)$ is guaranteed to have an inverse. Given a point $\mathbf{x} \in S_N$ we can find the α -parameters:

$$\alpha_i = |\Delta\mathbf{X}(i)| \sum_{j=1}^N [J^{-1}]_{i,j} (\Delta\mathbf{x} \cdot \mathbf{e}_j) \quad (8)$$

B. Volume Element

We can express the volume element of the simplex in terms of the α parameters:

$$\begin{aligned} d^N \mathbf{x} &= \left| \frac{\partial (\Delta\mathbf{x} \cdot \mathbf{e}(1), \dots, \Delta\mathbf{x} \cdot \mathbf{e}(N))}{\partial (\alpha(1) \dots \alpha(N))} \right| d\alpha_1 \dots d\alpha_N \\ &= \det[\mathbf{J}] \prod_{i=1}^N \frac{d\alpha_i}{|\Delta\mathbf{X}(i)|} \end{aligned} \quad (9)$$

The volume of the simplex is then

$$\begin{aligned} \text{Simplex Volume} &= \iint_{S_N} d^N \mathbf{x} = \iint_0^1 \det[\mathbb{J}] \prod_{i=1}^N \frac{d\alpha_i}{|\Delta \mathbf{X}(i)|} \\ &= \det[\mathbb{J}] \prod_{i=1}^N |\Delta \mathbf{X}(i)|^{-1}. \end{aligned} \quad (10)$$

C. Linear Interpolation

There is a unique linear interpolation function $\bar{f}(\mathbf{x})$ defined $\forall \mathbf{x} \in S_N$ such that

$$\bar{f}(\mathbf{X}_i) = f_i \quad \text{for } i = 0, 1, \dots, N \quad (11)$$

Over the entire tessellation, $\bar{f}$ is defined piece-wise for each N -simplex and will be continuous at the $N - 1$ dimensional boundaries, but will have discontinuous slopes at the boundaries.

D. Gradient

The first step in the construction of the interpolation is to compute the unique gradient defined by the $N + 1$ values f_i . For an arbitrary point $\mathbf{x}$ in the simplex,

$$\begin{aligned} \bar{f}(\mathbf{x}) &= f_0 + \mathbf{G} \cdot [\mathbf{x} - \mathbf{X}_0], \quad \text{with} \\ \mathbf{G} &\equiv \nabla \bar{f}(\mathbf{x}). \end{aligned} \quad (12)$$

$\mathbf{G}$ is the N -dimensional gradient of the approximating function $\bar{f}$.

The conditions of Eqs. 7 and 11 define $\mathbf{G}$ as follows. Start with the N linearly independent equations

$$\begin{aligned} \Delta f_i &= f_i - f_0 \quad i = 1, 2, \dots, N \\ \Delta f_i &= [\Delta \mathbf{X}_i] \cdot \mathbf{G} \\ &= J(i, j) [\mathbf{e}(j) \cdot \mathbf{G}] \end{aligned} \quad (13)$$

We already have the inverse matrix J^{-1} . Therefore we obtain the vector $\mathbf{G}$:

$$\mathbf{G} = \sum_{i,j} \mathbf{e}(j) [J^{-1}]_{j,i} \Delta f_i \quad (14)$$

III. CONCLUSIONS

These notes describe how to construct the interpolation function over the volume of each simplex. For a tessellations with m simplices in N dimesions, this requires the computation

of m matrices of dimension $N \times N$ and the inversion of each matrix. This is somewhat computationally intensive, although it only has to be done once. It will be important to establish if the tessellation algorithm itself generates either the vector $\mathbf{G}$, the matrix $\mathbb{J}$, and or either the inverse $\mathbb{J}^{-1}$ or determinant $|\mathbb{J}|$.