• The Gamma Function \( \Gamma (x) \)

For positive values of \( x \), the Gamma function is defined in terms of the improper integral,

\[
\Gamma (x) = \int_0^\infty t^{x-1}e^{-t}dt \quad x > 0.
\]

To ensure that the Gamma function is well defined, we must check to see that the improper integral converges for \( x > 0 \). Now,

\[
\int_0^\infty t^{x-1}e^{-t}dt = \int_0^1 t^{x-1}e^{-t}dt + \int_1^\infty t^{x-1}e^{-t}dt = I_1 + I_2
\]

\( I_1 \) is improper if \( x < 1 \), but for \( 0 \leq t \leq 1, 0 < e^{-t} \leq 1 \Rightarrow 0 \leq t^{x-1}e^{-t} \leq t^{x-1} \). Therefore,

\[
I_1 = \lim_{\varepsilon \to 0^+} \int_{\varepsilon}^{1} t^{x-1}e^{-t}dt \leq \lim_{\varepsilon \to 0^+} \int_{\varepsilon}^{1} \frac{e^{-t}}{x}dt = \lim_{\varepsilon \to 0^+} \frac{1}{x} \left(1 - \frac{1}{x^x}\right) = \frac{1}{x} \text{ if } x > 0. \text{ Thus } I_1 \text{ converges.}
\]

For \( I_2 \), if we let \( f(t) = t^{x+1}e^{-t} \Rightarrow f'(t) = e^{-t} [(x + 1)t^x - t^{x+1}] \)

\[
= t^x e^{-t} [x + 1 - t] \equiv 0 \text{ if } t = x + 1.
\]

\[
\begin{array}{c|c|c|c}
 t & x & x + 1 & x + 2 \\
 \hline
 f(t) & + & 0 & - \\
 f'(t) & \nearrow & \rightarrow & \searrow
\end{array}
\]

By the first derivative test, we see that \( f \) is maximized at \( t = x + 1 \). So,

\[
I_2 = \int_1^\infty t^{x-1}e^{-t}dt = \int_1^\infty f(t) \frac{d}{dt}dt \leq f(x + 1) \int_1^\infty \frac{1}{t}dt = f(x + 1). \text{ This shows that } I_2 \text{ also converges.}
\]

We have now shown that the Gamma function \( \Gamma(x) \) is well defined for \( x > 0 \) by

\[
\Gamma (x) = \int_0^\infty t^{x-1}e^{-t}dt \quad x > 0.
\]


1. \( \Gamma (x) > 0 \) for \( x > 0 \).
2. \( \Gamma (1) = 1, \Gamma (2) = 1, \Gamma (3) = 2; \Gamma (n) = (n - 1)! \) for \( n \in N \).
3. \( \Gamma (x) = (x - 1) \Gamma (x - 1) \) for \( x > 1 \).
4. \( \Gamma \left( \frac{1}{2} \right) = \sqrt{\pi} \)

Proofs: 1. Integrand is positive. 2., 3. Integrate by parts. 4. Let \( t = u^2 \) or see later.

**Example 1** \( \int_0^\infty x^4e^{-3x}dx \quad = \quad \frac{1}{3} \int_0^\infty \left( \frac{u}{3} \right)^4 e^{-u}du = \frac{1}{3} \Gamma (5) = \frac{4!}{3^4} = \frac{8}{8!} \)

\[
\uparrow \quad u = 3x
\]

**Example 2** \( \int_0^\infty x^2e^{-\frac{x^2}{2}}dx \quad = \quad \int_0^\infty \sqrt{2}u^{1/2}e^{-u}du = \sqrt{2} \Gamma (3/2) = \frac{\sqrt{2}}{2} \Gamma (1/2) = \sqrt{\pi} \)

\[
\uparrow \quad u = \frac{x^2}{2}
\]
Example 3 \( \int_0^\infty x^4 e^{-x^2} \, dx = \frac{1}{2} \int_0^\infty u^{2/3} e^{-u} \, du = \frac{1}{2} \Gamma \left( \frac{5}{3} \right) \) (Maple has this function built in—just like \( \exp \) or \( \sin \) or tangent.)

- The Beta Function \( B(m, n) \).

Although the integral \( \int_0^1 x^{m-1} (1-x)^{n-1} \, dx \) is improper at \( x = 0 \) if \( m < 1 \) and improper at \( x = 1 \) if \( n < 1 \), by considering the integral in the form

\[
\int_0^1 x^{m-1} (1-x)^{n-1} \, dx = \int_0^a + \int_a^b + \int_b^1 x^{m-1} (1-x)^{n-1} \, dx
\]

where \( 0 < a < b < 1 \), the improper integrals will converge if \( m, n > 0 \). Therefore, we define the Beta function \( B(m, n) \) by

\[
B(m, n) = \int_0^1 x^{m-1} (1-x)^{n-1} \, dx, \quad m, n > 0.
\]

By making the change of variable \( y = 1 - x \), we can see that \( B(m, n) = B(n, m) \). An alternative form for the Beta function is obtained via the substitution \( x = \sin^2 \theta \Rightarrow

\[
B(m, n) = 2 \int_0^{\pi/2} \sin^{2m-1} \theta \cos^{2n-1} \theta \, d\theta, \quad m, n > 0.
\]

Relationship between the Beta and Gamma Functions.

The very important result that relates the Beta and Gamma Functions is:

\[
B(m, n) = \frac{\Gamma(m) \Gamma(n)}{\Gamma(m+n)} \quad \text{for any } m, n > 0
\]

Proof:

Consider the double integral \( I = \iint_A f(x, y) \, dA \) where \( f(x, y) = x^{m-1} y^{n-1} e^{-(x^2+y^2)} \) and \( A \) is the first quadrant \( A = \{(x, y) : x > 0, y > 0\} \). Then, \( I = \left( \int_0^\infty x^{2m-1} e^{-x^2} \, dx \right) \left( \int_0^\infty y^{2n-1} e^{-y^2} \, dy \right) \).

Let \( x = \sqrt{7} \), then \( \int_0^\infty x^{2m-1} e^{-x^2} \, dx = \frac{1}{2} \int_0^\infty t^{m-1} e^{-t} \, dt = \frac{1}{2} \Gamma(m) \).

Similarly, \( \int_0^\infty y^{2n-1} e^{-y^2} \, dy = \frac{1}{2} \Gamma(n) \).

Thus,

\[
I = \frac{1}{4} \Gamma(m) \Gamma(n).
\]

Alternatively, if we switch to polar coordinates, we find

\[
I = \int_0^{\pi/2} \int_0^\infty (r \cos \theta)^{2m-1} (r \sin \theta)^{2n-1} e^{-r^2} \, r \, dr \, d\theta
= \left( \int_0^{\pi/2} \sin^{2m-1} \theta \cos^{2n-1} \theta \, d\theta \right) \left( \int_0^\infty r^{2m+2n-2} e^{-r^2} \, dr \right) = \left( \frac{1}{4} B(m, n) \right) \left( \frac{1}{2} \Gamma(m+n) \right), \quad \text{i.e.}
\]

\[
I = \frac{1}{4} B(m, n) \Gamma(m+n).
\]

Equating these results gives the result:

\[
B(m, n) = \frac{\Gamma(m) \Gamma(n)}{\Gamma(m+n)}
\]
Example 4 If we put \( m = n = 1/2 \) in trig. form of Beta function, we get
\[
B \left( \frac{1}{2}, 1/2 \right) = 2 \int_0^{\pi/2} \sin^{1-1}(\theta) \cos^{0} \theta d\theta = 2 \int_0^{\pi/2} \sin^0 \theta d\theta = \pi \Rightarrow \left\{ \Gamma \left( \frac{1}{2} \right) \right\}^2 = \Gamma \left( \frac{1}{2} \right) B \left( 1/2, 1/2 \right) = \pi \\
\Rightarrow \Gamma \left( \frac{1}{2} \right) = \sqrt{\pi}
\]

The trigonometric form of the Beta function, and the relationship with the Gamma functions allows us to evaluate integrals of the form \( \int_0^{\pi/2} \sin^m \theta \cos^n \theta d\theta \) without integrating. For example,

Example 5 \( \int_0^{\pi/2} \sin^3 \theta \cos^4 \theta d\theta = \frac{1}{2} \left( \frac{1}{2} \right) \frac{\Gamma \left( 2 \right) \Gamma \left( \frac{5}{2} \right)}{\Gamma \left( \frac{9}{2} \right)} = \frac{1}{2} \frac{\Gamma \left( 2 \right) \Gamma \left( \frac{5}{2} \right)}{\Gamma \left( 2 \right) \Gamma \left( \frac{5}{2} \right)} = \frac{2}{35} \)

Using the symmetry in the functions \( \sin \) and \( \cos \) over the four quadrants, we can actually use the above result for integrals over an interval which is a multiple of \([0, \pi/2]\). As an example, look at how \( \sin^3 x \) and \( \cos^4 x \) look like over \([0, 2\pi]\).

![Graph of sin^3 x and cos^4 x](image)

Because of trigonometric properties \( (\sin (\pi - x) = \sin x, \cos (\pi - x) = -\cos x, \text{ etc.}) \) the function \( \sin^3 x \cos^4 x \) possesses symmetries (as can be seen graphically) so that \( \int_0^{\pi/2} \sin^3 x \cos^4 x dx = \int_{\pi/2}^{\pi} \sin^3 x \cos^4 x dx = -\int_0^{3\pi/2} \sin^3 x \cos^4 x dx = -\int_{3\pi/2}^{2\pi} \sin^3 x \cos^4 x dx \). The sign of the integrand in each quadrant is simply the product of the signs of \( \sin^3 x \) and \( \cos^4 x \) over the same quadrants. Once these signs have been determined, we can express a definite integral over several quadrants as a multiple of the integral over the first quadrant. For example,

Example 6 \( \int_0^{\pi} \sin^3 x \cos^4 x dx = (1 + 1) \int_0^{\pi/2} \sin^3 x \cos^4 x dx = 2 \left( \frac{2}{35} \right) = \frac{4}{35} \)

Example 7 \( \int_{3\pi/2}^{2\pi} \sin^3 x \cos^4 x dx = (1 + 1 - 1) \int_0^{\pi/2} \sin^3 x \cos^4 x dx = 1 \left( \frac{2}{35} \right) = \frac{2}{35} \)

Example 8 \( \int_{3\pi/2}^{2\pi} \sin^3 x \cos^4 x dx = (1 + 1 - 1) \int_0^{\pi/2} \sin^3 x \cos^4 x dx = 0 \)

The procedure outlined here applies to all integrals of the form
\[
\int_{i\pi/2}^{j\pi/2} \sin^m x \cos^n x dx
\]

where \( i, j, m, n \) are integers and \( m, n \geq 0 \). Since all we need is the sign of the integrand, we do not need to sketch \( \sin^m x \cos^n x \); a simple table of signs will be sufficient - as seen in this example.
Example 9 \( \int_0^{3\pi/2} \sin^2 x \cos^5 x \, dx = K \int_0^{\pi/2} \sin^2 x \cos^5 x \, dx \). To obtain \( K \), we see that

<table>
<thead>
<tr>
<th>( x )</th>
<th>( 0, \pi/2 )</th>
<th>( \pi/2, \pi )</th>
<th>( \pi, 3\pi/2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \sin^2 x )</td>
<td>+</td>
<td>+</td>
<td>+</td>
</tr>
<tr>
<td>( \cos^5 x )</td>
<td>+</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>( \sin^2 x \cos^5 x )</td>
<td>+</td>
<td>-</td>
<td>-</td>
</tr>
</tbody>
</table>

\[ \Rightarrow K = (1 - 1) = -1. \text{ Therefore,} \]

\[ \int_0^{3\pi/2} \sin^2 x \cos^5 x \, dx = -\int_0^{\pi/2} \sin^2 x \cos^5 x \, dx = -\frac{1}{2} B \left( \frac{3}{2}, 3 \right) = -\frac{1}{2} \frac{\Gamma \left( \frac{3}{2} \right) \Gamma (3)}{\Gamma \left( \frac{3}{2} \right)} = -\frac{8}{105}. \]

It must be stated that the procedure outlined does not apply to integrals like \( \int_0^{\pi/3} \sin^2 x \cos^5 x \, dx \) - a non integer multiple of \( \pi/2 \). In this case, the usual integration techniques must be used.

Exercises

1. Evaluate the following:
   - (a) \( \Gamma (6) : \frac{120}{1} \)
   - (b) \( \Gamma (3/2) : \frac{1}{\sqrt{\pi}} \)
   - (c) \( \frac{\Gamma (9/2)}{\Gamma (5/2)} : \frac{35}{12} \)
   - (d) \( B (3, 2) : \frac{1}{12} \)
   - (e) \( B (4, 1/2) : \frac{32}{35} \)

2. Use symmetry and the Beta function to evaluate:
   - (a) \( \int_0^{\pi/2} \sin^3 x \cos^2 x \, dx : \frac{2}{15} \)
   - (b) \( \int_0^{\pi} \sin^3 x \cos^2 x \, dx : \frac{4}{15} \)
   - (c) \( \int_0^{\pi} \sin^5 x \, dx : \frac{16}{15} \)
   - (d) \( \int_0^{\pi} \sin^7 x \cos^3 x \, dx : 0 \)
   - (e) \( \int_0^{\pi} \cos^5 x \, dx : 0 \)
   - (f) \( \int_0^{2\pi} \sin^5 x \cos^2 x \, dx : 0 \)
   - (g) \( \int_0^{2\pi} \sin^4 x \cos^2 x \, dx : \frac{1}{8\pi} \)