Linear Combinations

A vector $\vec{b}$ in $\mathbb{R}^m$ is called a linear combination of the vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ in $\mathbb{R}^m$ if there are scalars $c_1, c_1, \ldots, c_n$, such that

$$c_1 \vec{v}_1 + c_2 \vec{v}_2 + \cdots + c_n \vec{v}_n = \vec{b}$$

for example, the vector

$$\begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

in $\mathbb{R}^2$ is a linear combination of the vectors

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

in $\mathbb{R}^2$ because

$$\begin{bmatrix} 2 \\ 3 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 3 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$
and (a not so obvious example), the vector

\[
\begin{bmatrix}
7 \\
4 \\
7
\end{bmatrix}
\]

in \( \mathbb{R}^3 \) is a linear combination of the vectors

\[
\begin{bmatrix}
3 \\
1 \\
2
\end{bmatrix}
and
\begin{bmatrix}
0 \\
4 \\
5
\end{bmatrix}
and
\begin{bmatrix}
2 \\
0 \\
1
\end{bmatrix}
\]

in \( \mathbb{R}^3 \) because

\[
\begin{bmatrix}
7 \\
4 \\
7
\end{bmatrix} = 2 \begin{bmatrix}
3 \\
1 \\
2
\end{bmatrix} + \frac{1}{2} \begin{bmatrix}
0 \\
4 \\
5
\end{bmatrix} + \frac{1}{2} \begin{bmatrix}
2 \\
0 \\
1
\end{bmatrix}
\]

**Linear Relations**

Consider the vectors \( \vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n \) in \( \mathbb{R}^m \). An equation of the form

\[
c_1 \vec{v}_1 + c_2 \vec{v}_2 + \cdots + c_n \vec{v}_n = \vec{0}
\]

is called a linear relation among the vectors \( \vec{v}_i \).
There is always the “trivial” relation, where $c_1 = c_2 = \cdots = c_n = 0$.

“Non-trivial” relations may or may not exist among the vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$.

**Linear Dependence**

The vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ in $\mathbb{R}^m$ are called linearly dependent if one of the vectors can be expressed as a linear combination of the others.

Claim: If there is a nontrivial relation among the vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ in $\mathbb{R}^m$, then the vectors are linearly dependent.

Proof: If there is a nontrivial relation among the vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ in $\mathbb{R}^m$, then

$$c_1 \vec{v}_1 + c_2 \vec{v}_2 + \cdots + c_n \vec{v}_n = \vec{0}, \quad \text{with} \quad c_i \neq 0$$

we can solve for $\vec{v}_i$ in terms of the other vectors and thus express $\vec{v}_i$ as a linear combination of the other vectors. If any vector can be expressed as a linear combination of the other vectors, then the set of vectors is linearly dependent.

**Linear Independence**

The vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ in $\mathbb{R}^m$ are called linearly independent if none of the vectors can be expressed as a linear combination of the others.
It follows from the claim and proof above that a set of vectors \( \vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n \) in \( \mathbb{R}^m \) is linearly independent only if no nontrivial relations exist among them. Or, the set of vectors is linearly independent if the only linear relation

\[
c_1 \vec{v}_1 + c_2 \vec{v}_2 + \cdots + c_n \vec{v}_n = \vec{0}
\]

among them is the trivial relation, where \( c_1 = c_2 = \cdots = c_n = 0 \).

We can also think of this linear relation

\[
c_1 \vec{v}_1 + c_2 \vec{v}_2 + \cdots + c_n \vec{v}_n = \vec{0}
\]

as a system of \( m \) equations in \( n \) unknowns, or

\[
\begin{bmatrix}
\vec{v}_1 \\
\vec{v}_2 \\
\vdots \\
\vec{v}_n
\end{bmatrix}
\begin{bmatrix}
c_1 \\
c_2 \\
\vdots \\
c_n
\end{bmatrix}
= \vec{0}
\]

or

\[
A \bar{c} = \vec{0}
\]

where \( A \) is the \( m \times n \) matrix whose columns are the vectors \( \vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n \) in \( \mathbb{R}^m \).

This says that a set vectors \( \vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n \) in \( \mathbb{R}^m \) is linearly independent only if the solution to the system \( A \bar{c} = \vec{0} \) is

\[
\begin{bmatrix}
c_1 \\
c_2 \\
\vdots \\
c_n
\end{bmatrix}
= \begin{bmatrix}
0 \\
0 \\
\vdots \\
0
\end{bmatrix}
\]
In augmented matrix form, this says that a set vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ in $\mathbb{R}^m$ is linearly independent only if the solution to the following system is $\vec{0}$.

\[
\begin{bmatrix}
| & | & | & : & 0 \\
\vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n & : \\
| & | & | & : & 0 
\end{bmatrix}
\]

Since the system above can never contain a row of $[0 \ 0 \ \cdots \ 0 : 1]$, it will be consistent and must have either a unique solution or an infinite number of solutions. For the vectors to be linearly independent, the system must have the unique solution $\vec{0}$.

A consistent system will have a unique solution only if there are no nonleading variables. Therefore, this system will have the unique solution of $\vec{0}$ only if there are no nonleading variables, or if the rank $= n$.

\[\therefore \text{ The set of vectors } \vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n \text{ in } \mathbb{R}^m \text{ is linearly independent only if the rank}(A) = n, \text{ where} \]

\[A = \begin{bmatrix}
| & | & | & | \\
\vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \\
| & | & | & |
\end{bmatrix}\]
Span

The set of all linear combinations of the vectors \( \vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n \) in \( \mathbb{R}^m \) is called their span:

\[
\text{span}(\vec{v}_1, \ldots, \vec{v}_n) = \{ c_1 \vec{v}_1 + \cdots + c_n \vec{v}_n : c_i \text{ arbitrary scalars} \}
\]

If the span of the set of vectors \( \vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n \) in \( \mathbb{R}^m \) is \( \mathbb{R}^m \), then we would say the set of vectors “spans” \( \mathbb{R}^m \).

Given a set of vectors \( \vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n \) in \( \mathbb{R}^m \), consider the question, does this set of vectors span \( \mathbb{R}^m \)?

If this set of vectors spans \( \mathbb{R}^m \), then by the definition of span, for all vectors \( \vec{y} \) in \( \mathbb{R}^m \), there exists a set of scalars \( \{ c_1, c_2, \ldots, c_n \} \) such that

\[
c_1 \vec{v}_1 + c_2 \vec{v}_2 + \cdots + c_n \vec{v}_n = \vec{y}
\]

This is equivalent to saying that for all vectors \( \vec{y} \) in \( \mathbb{R}^m \), there exists a solution to the following system of linear equations

\[
\begin{bmatrix}
\vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n
\end{bmatrix}
\begin{bmatrix}
c_1 \\
c_2 \\
\vdots \\
c_n
\end{bmatrix}
= \begin{bmatrix}
y_1 \\
y_2 \\
\vdots \\
y_m
\end{bmatrix}
\]

which can be represented by the following matrix
The system above must have at least one solution, or, its rref may not contain a row of $[0 \ 0 \ \cdots \ 0 \ : \ 1]$. But since this must hold for all $\vec{y}$, there will exist a nonzero right hand side and thus a row of all 0’s in the coefficient side of the rref will lead to an inconsistent system, or no solutions.

Therefore, a solution to the system above will exist for all $\vec{y}$ in $\mathbb{R}^m$ only if the rref does not contain a row of all 0’s on the coefficient side. This is the same as saying that the rank of the coefficient matrix must equal $m$.

\[ \therefore \text{The set of vectors } \vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n \text{ in } \mathbb{R}^m \text{ spans } \mathbb{R}^m \text{ only if the rank}(A) = m, \]

where

\[ A = \begin{bmatrix} | & | & \cdots & | \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \end{bmatrix} \]

In summary, given a set of vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ in $\mathbb{R}^m$ and the $m \times n$ matrix $A$ whose columns are the vectors $\vec{v}_i$,

the vectors are linearly independent in $\mathbb{R}^m$ if rank($A$) = $n$

the vectors span $\mathbb{R}^m$ if rank($A$) = $m$