Inverse of a Linear Transformation

Invertible functions — a function $f$ from $X$ to $Y$ is called invertible if the equation $y = f(x)$ has a unique solution $x$ in $X$ for each $y$ in $Y$.

Consider the following example, where $f$, $g$, and $h$ are functions from the finite set $X$ to the finite set $Y$:

- $f$ is invertible
- $g$ is not invertible: The equation $y_2 = g(x)$ has two solutions, $x_2$ and $x_3$
- $h$ is not invertible: There is no $x$ such that $y_3 = h(x)$

If a function $f$ from $X$ to $Y$ is invertible, then its inverse $f^{-1}$ from $Y$ to $X$ is defined by $x = f^{-1}(y) = \text{(the unique } x \text{ in } X \text{ such that } y = f(x))$. 
Now consider the case of a linear transformation (a function involving vectors instead of scalars) from $\mathbb{R}^n$ to $\mathbb{R}^m$ given by $\vec{y} = A\vec{x}$ where $\vec{x} \in \mathbb{R}^n$, $\vec{y} \in \mathbb{R}^m$, and $A$ is an $m \times n$ matrix.

**Invertible Linear Transformations** — the linear transformation $\vec{y} = A\vec{x}$ is invertible if the linear system $A\vec{x} = \vec{y}$ has a unique solution $\vec{x} \in \mathbb{R}^n$ for all $\vec{y} \in \mathbb{R}^m$.

Consider the following example, where $T$, $R$, and $S$ are linear transformations from the $\mathbb{R}^n$ to $\mathbb{R}^m$:

$\vec{y} = T(\vec{x})$

$\vec{y} = R(\vec{x})$

$\vec{y} = S(\vec{x})$

$T$ is invertible

$R$ is not invertible: The equation $\vec{y}_2 = R(\vec{x})$ has two solutions, $\vec{x}_2$ and $\vec{x}_3$

$S$ is not invertible: There is no $\vec{x}$ such that $\vec{y}_3 = S(\vec{x})$

If a linear transformation $T$ from $\mathbb{R}^n$ to $\mathbb{R}^m$ is invertible, then its inverse $T^{-1}$ from $\mathbb{R}^m$ to $\mathbb{R}^n$ is defined by $\vec{x} = T^{-1}(\vec{y}) = \text{(the unique } \vec{x} \text{ in } \mathbb{R}^n \text{ such that } \vec{y} = T(\vec{x}))$. 
A matrix $A$ is called invertible if the linear transformation $\tilde{y} = A\tilde{x}$ is invertible. The matrix of the inverse transformation is denoted by $A^{-1}$. If the transformation $\tilde{y} = A\tilde{x}$ is invertible, its inverse is $\tilde{x} = A^{-1}\tilde{y}$.

The transformation $\tilde{y} = A\tilde{x}$ is invertible if the linear system

$$A\tilde{x} = \tilde{y}$$

has a unique solution $\tilde{x}$ for all $\tilde{y} \in \mathbb{R}^m$.

When does the system $A\tilde{x} = \tilde{y}$ have a unique solution, where $A$ is an $m \times n$ matrix?

Consider all the cases of $A$: $m < n$, $m = n$, and $m > n$:

$m < n$: The system $A\tilde{x} = \tilde{y}$ has fewer equations than unknowns, which means there must be a nonleading variable, which means that the system has either no solutions or infinitely many solutions.

$m = n$: The system $A\tilde{x} = \tilde{y}$ has a unique solution if and only if $\text{rref}(A) = I_n$.

$m > n$: We can find a vector $\tilde{y} \in \mathbb{R}^m$ such that the system is inconsistent, therefore the system does not have a unique solution $\tilde{x} \in \mathbb{R}^n$ for all $\tilde{y} \in \mathbb{R}^m$.

Therefore, an $m \times n$ matrix $A$ is invertible if and only if

a. $A$ is a square matrix (i.e., $m = n$), and

b. $\text{rref}(A) = I_n$. 
If the matrix $A$ is invertible, how can we find the inverse matrix $A^{-1}$?

Consider the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 2 \\ 3 & 8 & 2 \end{bmatrix}$$

or, equivalently, the linear transformation $\vec{y} = A\vec{x}$

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 + x_3 \\ 2x_1 + 3x_2 + 2x_3 \\ 3x_1 + 8x_2 + 2x_3 \end{bmatrix}$$

To find the inverse transformation, we solve this system for the input variables $x_1$, $x_2$, and $x_3$ in terms of $y_1$, $y_2$, and $y_3$:

$$\begin{align*}
x_1 + x_2 + x_3 &= y_1 \\
2x_1 + 3x_2 + 2x_3 &= y_2 \\
3x_1 + 8x_2 + 2x_3 &= y_3
\end{align*}$$

We can write this in matrix form (and solve) as:

$$\begin{bmatrix}
1 & 1 & 1 & 1 & 0 & 0 \\
2 & 3 & 2 & 0 & 1 & 0 \\
3 & 8 & 2 & 0 & 0 & 1
\end{bmatrix} \rightarrow \begin{bmatrix}
1 & 1 & 1 & 1 & 0 & 0 \\
0 & 1 & 0 & -2 & 1 & 0 \\
0 & 5 & -1 & -3 & 0 & 1
\end{bmatrix} \rightarrow \begin{bmatrix}
1 & 0 & 1 & 3 & -1 & 0 \\
0 & 1 & 0 & -2 & 1 & 0 \\
0 & 0 & -1 & 7 & -5 & 1
\end{bmatrix} \rightarrow \begin{bmatrix}
1 & 0 & 1 & 10 & -6 & 1 \\
0 & 1 & 0 & -2 & 1 & 0 \\
0 & 0 & 1 & -7 & 5 & -1
\end{bmatrix}.$$
which says that

\[
\begin{align*}
  x_1 &= 10y_1 - 6y_2 + y_3 \\
  x_2 &= -2y_1 + y_2 \\
  x_3 &= -7y_1 + 5y_2 - y_3
\end{align*}
\]

or, equivalently,

\[
\begin{bmatrix}
  x_1 \\
  x_2 \\
  x_3
\end{bmatrix} =
\begin{bmatrix}
  10y_1 - 6y_2 + y_3 \\
  -2y_1 + y_2 \\
  -7y_1 + 5y_2 - y_3
\end{bmatrix}
\]

To find the inverse of an \( n \times n \) matrix \( A \):

Form the \( n \times (2n) \) matrix \([A:I_n]\) and compute \( \text{rref}[A:I_n] \).

- If \( \text{rref}[A:I_n] \) is of the form \([I_n:B]\), then \( A \) is invertible, and \( A^{-1} = B \).

- If \( \text{rref}[A:I_n] \) is of another form (i.e., its left half fails to be \( I_n \)), then \( A \) is not invertible. (Note that the left half of \( \text{rref}[A:I_n] \) is \( \text{rref}(A) \).
Inverse and determinant of a $2 \times 2$ matrix

a. The $2 \times 2$ matrix

\[ A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \]

is invertible if (and only if) $ad - bc \neq 0$

The quantity $ad - bc$ is called the determinant of $A$, written $\det(A)$.

b. If

\[ A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \]

is invertible, then

\[
\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{\det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}
\]