Basis of the Image and Kernel of a Linear Transformation

Suppose we are given a linear transformation \( T \) from \( \mathbb{R}^n \) to \( \mathbb{R}^m \). If we can find a set of vectors that forms a basis of \( \text{im}(T) \), then we know that \( \text{im}(T) \) is all possible linear combinations of those basis vectors. The number of vectors in the basis of \( \text{im}(T) \), or \( \dim(\text{im}(T)) \), will give us a better picture of the image itself.

For example, suppose that \( T \) is a linear transformation from \( \mathbb{R}^3 \) to \( \mathbb{R}^3 \). If we find that the basis of \( \text{im}(T) \) consists of one vector, then the image is simply a line in \( \mathbb{R}^3 \) (any scalar multiple of a single vector in \( \mathbb{R}^3 \)). If we find that the basis of \( \text{im}(T) \) consists of two vectors, then the image is simply a plane in \( \mathbb{R}^3 \) (any linear combination of two vectors in \( \mathbb{R}^3 \)). If we find that the basis of \( \text{im}(T) \) consists of three vectors, then the image is the entire codomain, or all of \( \mathbb{R}^3 \) (any linear combination of three vectors that form a basis of \( \mathbb{R}^3 \)).

The same is true for the \( \ker(T) \). That is, if we can find a set of vectors that forms a basis of \( \ker(T) \), then we know that \( \ker(T) \) is all possible linear combinations of those basis vectors. The number of vectors in the basis of \( \ker(T) \), or \( \dim(\ker(T)) \), will give us a better picture of the kernel itself.

For example, suppose that \( T \) is a linear transformation from \( \mathbb{R}^3 \) to \( \mathbb{R}^3 \). If we find that the basis of \( \ker(T) \) consists of one vector, then the kernel is simply a line in \( \mathbb{R}^3 \) (any scalar multiple of a single vector in \( \mathbb{R}^3 \)). If we find that the basis of \( \ker(T) \) consists of two vectors, then the kernel is simply a plane in \( \mathbb{R}^3 \) (any linear combination of two vectors in \( \mathbb{R}^3 \)). If we find that the basis of \( \ker(T) \) consists of three vectors, then the kernel is the entire domain, or all of \( \mathbb{R}^3 \) (any linear combination of three vectors that form a basis of \( \mathbb{R}^3 \)).
Finding Bases for the Image and Kernel of a Linear Transformation

There is a systematic way to find a set of vectors that forms a basis of $\text{im}(T)$ and a set of vectors that forms a basis of $\text{ker}(T)$, where $T$ is a linear transformation given by $T(\vec{x}) = A\vec{x}$. In this method, we will interchange the idea of the image and kernel of a linear transformation with the idea of the image and kernel of a matrix; or, we assume that $\text{im}(A) \equiv \text{im}(T)$ and $\text{ker}(A) \equiv \text{ker}(T)$.

To find the bases for the image and kernel of a matrix $A$:

1. Consider the system $A\vec{x} = \vec{0}$.
2. Use elimination to reduce the system to rref.
3. Identify the leading 1’s.
4. The columns in $A$ corresponding to the columns in $\text{rref}(A)$ with leading 1’s form a basis of $\text{im}(A)$.
5. If the solution to the system is unique (the vector $\vec{0}$), then $\text{ker}(A) = \vec{0}$, there are no vectors in the basis of $\text{ker}(A)$, and thus $\text{dim}(\text{ker}(A)) = 0$.
6. If there are an infinite number of solutions to the system (remember, either unique solution or infinite number of solutions when the right hand side is all 0’s), write the solution of the system as a linear combination of arbitrary variables and vectors. The vectors in the solution form a basis of $\text{ker}(A)$.
Example:

Given the linear transformation $T$ from $\mathbb{R}^4$ to $\mathbb{R}^4$ given by $T(\tilde{x}) = A\tilde{x}$, where

$$A = \begin{bmatrix}
1 & -4 & -2 & 3 \\
1 & -4 & 1 & 6 \\
-1 & 3 & 2 & -1 \\
1 & -1 & -2 & -3
\end{bmatrix}$$

find a basis for the $\text{im}(T)$ and a basis for the $\ker(T)$.

First write the system $A\tilde{x} = \tilde{0}$ in augmented matrix form and use elimination to reduce the system to rref:

$$\begin{bmatrix}
1 & -4 & -2 & 3 & : & 0 \\
1 & -4 & 1 & 6 & : & 0 \\
-1 & 3 & 2 & -1 & : & 0 \\
1 & -1 & -2 & -3 & : & 0
\end{bmatrix} \rightarrow \begin{bmatrix}
1 & 0 & 0 & -3 & : & 0 \\
0 & 1 & 0 & -2 & : & 0 \\
0 & 0 & 1 & 1 & : & 0 \\
0 & 0 & 0 & 0 & : & 0
\end{bmatrix}$$

The first three columns of $\text{rref}(A)$ have leading 1’s $\Rightarrow \{\bar{v}_1, \bar{v}_2, \bar{v}_3\}$ forms a basis of $\text{im}(A)$, or

$$\begin{bmatrix}
1 \\
1 \\
-1 \\
1
\end{bmatrix}, \begin{bmatrix}
-4 \\
-4 \\
3 \\
-1
\end{bmatrix}, \begin{bmatrix}
-2 \\
1 \\
2 \\
-2
\end{bmatrix} = \text{basis of im}(A)$$

$$\Rightarrow \text{im}(A) = \text{all linear combinations of the basis vectors, or}$$

$$\text{im}(A) = c_1\bar{v}_1 + c_2\bar{v}_2 + c_3\bar{v}_3, \text{ where } c_1, c_2, c_3 \text{ are arbitrary.}$$
Now find the solution to the system:

Assume the fourth column corresponds to a variable $x_4$. This column does not have a leading 1, so it is nonleading. Therefore, let $x_4 = t$, where $t$ is arbitrary.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 3t \\ 2t \\ -t \\ t \end{bmatrix} = t \begin{bmatrix} 3 \\ 2 \\ -1 \\ 1 \end{bmatrix}, \quad t \text{ arbitrary}$$

$$\Rightarrow \left\{ \begin{bmatrix} 3 \\ 2 \\ -1 \\ 1 \end{bmatrix} \right\} \text{ forms a basis of } \ker(A)$$

$$\Rightarrow \ker(A) = \text{all linear combinations of the basis vectors, or}$$

$$\ker(A) = c_1 \begin{bmatrix} 3 \\ 2 \\ -1 \\ 1 \end{bmatrix}, \quad \text{where } c_1 \text{ is arbitrary.}$$
Rank-Nullity Theorem

Notice that the number of vectors in the basis of the \( \text{im}(A) \) will always be equal to the number of leading 1’s, which is just the \( \text{rank}(A) \), or

\[
\dim(\text{im}(A)) = \text{rank}(A)
\]

Also notice that the number of vectors in the basis of the \( \text{ker}(A) \) will always be equal to the number of nonleading variables, which will always be the number of columns of \( A \) minus the number of leading 1’s, or

\[
\dim(\text{ker}(A)) = n - \text{rank}(A)
\]

The dimension of the \( \text{ker}(A) \) is defined as the **nullity**.

\[
\therefore \quad \dim(\text{im}(A)) + \dim(\text{ker}(A)) = \text{rank}(A) + \text{nullity}(A) = \text{rank}(A) + n - \text{rank}(A), \text{ or }
\]

\[
\text{rank}(A) + \text{nullity}(A) = n \quad \text{(Rank-Nullity Theorem)},
\]

where \( n \) is the number of columns in \( A \).