Cramer’s Rule

Consider the general linear system of equations $A\vec{x} = \vec{b}$, where $A$ is and invertible $n \times n$ matrix. The components $x_i$ of the solution vector $\vec{x}$ are:

$$x_i = \frac{\det(A_i)}{\det(A)}$$

where $A_i$ is the matrix obtained by replacing the $i$'th column of $A$ by $\vec{b}$.

Proof of Cramer’s Rule:

$$A = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_i & \cdots & \vec{v}_n \end{bmatrix} \quad \text{(each } \vec{v}_i \in \mathbb{R}^n)$$

$$A_i = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{b} & \cdots & \vec{v}_n \end{bmatrix}$$

$$A_i = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & A\vec{x} & \cdots & \vec{v}_n \end{bmatrix}$$

$$A_i = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & (x_1\vec{v}_1 + x_2\vec{v}_2 + \ldots + x_n\vec{v}_n) & \cdots & \vec{v}_n \end{bmatrix}$$

$$\det(A_i) = \det \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & (x_1\vec{v}_1 + x_2\vec{v}_2 + \ldots + x_n\vec{v}_n) & \cdots & \vec{v}_n \end{bmatrix}$$

by the linearity of the determinant in the columns, we get

$$\det(A_i) = \det \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & x_1\vec{v}_1 & \cdots & \vec{v}_n \end{bmatrix}$$

$$+ \det \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & x_2\vec{v}_2 & \cdots & \vec{v}_n \end{bmatrix}$$

$$+ \ldots + \det \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & x_i\vec{v}_i & \cdots & \vec{v}_n \end{bmatrix}$$

$$+ \ldots + \det \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & x_n\vec{v}_n & \cdots & \vec{v}_n \end{bmatrix}$$
also by the linearity of the determinant in the columns

\[ \det(A_i) = x_1 \det \begin{bmatrix} \bar{v}_1 & \bar{v}_2 & \cdots & \bar{v}_i & \cdots & \bar{v}_n \end{bmatrix} \\
+ x_2 \det \begin{bmatrix} \bar{v}_1 & \bar{v}_2 & \cdots & \bar{v}_i & \cdots & \bar{v}_n \end{bmatrix} \\
+ \ldots + x_i \det \begin{bmatrix} \bar{v}_1 & \bar{v}_2 & \cdots & \bar{v}_i & \cdots & \bar{v}_n \end{bmatrix} \\
+ \ldots + x_n \det \begin{bmatrix} \bar{v}_1 & \bar{v}_2 & \cdots & \bar{v}_i & \cdots & \bar{v}_n \end{bmatrix} \]

all of the terms above are zero, except the term with \( \bar{v}_i \) in the \( i \)’th column (because all the rest will have two columns that are identical)

\[ \det(A_i) = x_i \det \begin{bmatrix} \bar{v}_1 & \bar{v}_2 & \cdots & \bar{v}_i & \cdots & \bar{v}_n \end{bmatrix} \]

\[ \det(A_i) = x_i \det(A) \Rightarrow x_i = \frac{\det(A_i)}{\det(A)} \]

Example

Solve the following system of linear equations using Cramer’s Rule.

\[
\begin{align*}
2x + 3y &= 7 \\
4y + 5z &= 19 \\
6x + 7z &= 33
\end{align*}
\]

\[
A = \begin{bmatrix} 2 & 3 & 0 \\
0 & 4 & 5 \\
6 & 0 & 7 \end{bmatrix} \Rightarrow \det(A) = 2(28) - 3(-30) + 0 = 146
\]
$A_1 = \begin{bmatrix} 7 & 3 & 0 \\ 19 & 4 & 5 \\ 33 & 0 & 7 \end{bmatrix}$ \quad \Rightarrow \quad \det(A_1) = 7(28) - 3[19(7) - 33(5)] = 292

$A_2 = \begin{bmatrix} 2 & 7 & 0 \\ 0 & 19 & 5 \\ 6 & 33 & 7 \end{bmatrix}$ \quad \Rightarrow \quad \det(A_2) = 2[19(7) - 33(5)] - 7(-30) = 142

$A_3 = \begin{bmatrix} 2 & 3 & 7 \\ 0 & 4 & 19 \\ 6 & 0 & 33 \end{bmatrix}$ \quad \Rightarrow \quad \det(A_3) = 2(4)(33) - 3(-6)(19) + 7(-24) = 438

\[
x_1 = \frac{\det(A_1)}{\det(A)} = \frac{292}{146} = 2
\]

\[
x_2 = \frac{\det(A_2)}{\det(A)} = \frac{146}{146} = 1
\]

\[
x_3 = \frac{\det(A_3)}{\det(A)} = \frac{438}{146} = 3
\]

Corollary to Cramer’s Rule

Consider an invertible $n \times n$ matrix $A$. The classical adjoint $\text{adj}(A)$ is the $n \times n$ matrix whose $ij$’th entry is

\[(-1)^{i+j} \det(A_{ji})\]

The classical adjoint can be used to find the inverse of a matrix by the formula

\[A^{-1} = \frac{1}{\det(A)} \text{adj}(A)\]