Diagonalization and Powers of a Matrix

Diagonalization of a Matrix

An $n \times n$ matrix A is called diagonalizable if A is similar to a diagonal matrix D; that is, if there is an invertible $n \times n$ matrix S such that $D = S^{-1}AS$ is diagonal.

A matrix A is diagonalizable iff there is an eigenbasis for A.

To determine whether a given $n \times n$ matrix is diagonalizable and, if so, to find an invertible matrix S such that $D = S^{-1}AS$ is diagonal:

a. Find the eigenvalues of A.

b. For each eigenvalue λ, find the eigenvectors.

c. If there are a total of n eigenvectors then construct the eigenbasis $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$, where the $\vec{v}_i$'s are the eigenvectors (if the total number of eigenvectors is $< n$, then the matrix is not diagonalizable).

d. Let $S = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \ldots & \vec{v}_n \end{bmatrix}$.

e. The matrix $D = S^{-1}AS$ is diagonal, with the corresponding eigenvalues on the diagonal.

Example

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \text{Is } A \text{ diagonalizable? If so, diagonalize } A$$

upper triangular matrix $\Rightarrow$ eigenvalues are the diagonal entries 0, 1

$$\begin{aligned} \lambda_1 &= 0 \quad \text{(with algebraic multiplicity of 2)} \\ \lambda_2 &= 1 \quad \text{(with algebraic multiplicity of 1)} \end{aligned}$$

$$E_0 = \ker\left(\begin{bmatrix} -1 & -1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right) = \ker\left(\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right)$$

$$E_0 = s\begin{bmatrix} -1 \\ 1 \\ \end{bmatrix}$$

Note: $D = S^{-1}AS$, but you don't need to actually compute $S^{-1}AS$ because we know that the diagonal entries of D are the eigenvalues of A. You just need to make sure that the order that the eigenvectors appear in S match the order that the eigenvalues appear in D.

can check by seeing if $AS = SD$ (that way you don't ever have to find S^{-1}):

$$\begin{aligned} AS &= \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} -1 & -1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ SD &= \begin{bmatrix} -1 & -1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{aligned}$$

<u>Powers of a Matrix</u>

Consider the matrix $A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$ and suppose you are asked to find A^5. Given what we know thus far, we would have to find this by a series of matrix-matrix multiplications; i.e.,

$$A^5 = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$$

Obviously, this would get very complicated and cumbersome.

There is an easier way! (in some cases)

Given a diagonalizable matrix A, we can compute the powers A^t (where t is a positive integer) as follows:

a. Diagonalize A; i.e., find an invertible S (using the eigenbasis) and the diagonal matrix D (which is simply the eigenvalues on the diagonal)

b. Then $D = S^{-1}AS$, or $A = SDS^{-1}$ and $\underline{A^t = SD^tS^{-1}}$

Proof of $A^t = SD^tS^{-1}$

$$\begin{aligned} \text{if} \quad A &= SDS^{-1} \\ A^t &= (SDS^{-1})^t \\ A^t &= (SDS^{-1})(SDS^{-1})\cdots(SDS^{-1}) \quad (t \text{ times}) \\ &\quad \text{all inner terms cancel out} \\ A^t &= SD^tS^{-1} \end{aligned}$$

Example

$$A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} \qquad \text{Find a formula for } A^t$$

$$\det \begin{bmatrix} \lambda - 1 & -2 \\ -4 & \lambda - 3 \end{bmatrix} = 0 \quad \Rightarrow \quad (\lambda-1)(\lambda-3) - 8 = \lambda^2 - 4\lambda + 3 - 8$$

$$= \lambda^2 - 4\lambda - 3 - 5 = (\lambda+1)(\lambda-5) = 0$$

$$\Rightarrow \quad \lambda = -1 \text{ or } \lambda = 5$$

$$E_{-1} = \ker\left(\begin{bmatrix} -2 & -2 \\ -4 & -4 \end{bmatrix}\right) = \ker\left(\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\right) = t\begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

where t is arbitrary

$$E_{5} = \ker\left(\begin{bmatrix} 4 & -2 \\ -4 & 2 \end{bmatrix}\right) = \ker\left(\begin{bmatrix} 1 & -1/2 \\ 0 & 0 \end{bmatrix}\right) = t\begin{bmatrix} 1/2 \\ 1 \end{bmatrix}$$

where t is arbitrary

$$\text{eigenbasis} = \left\{\begin{bmatrix} -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1/2 \\ 1 \end{bmatrix},\right\}$$

$$S = \begin{bmatrix} -1 & 1/2 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad S^{-1} = \begin{bmatrix} -2/3 & 1/3 \\ 2/3 & 2/3 \end{bmatrix}$$

$$D = \begin{bmatrix} -1 & 0 \\ 0 & 5 \end{bmatrix} \quad \text{and} \quad D^t = \begin{bmatrix} (-1)^t & 0 \\ 0 & 5^t \end{bmatrix}$$

$$A^t = SD^tS^{-1} = \begin{bmatrix} -1 & 1/2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} (-1)^t & 0 \\ 0 & 5^t \end{bmatrix}\begin{bmatrix} -2/3 & 1/3 \\ 2/3 & 2/3 \end{bmatrix}$$

$$A^t = \begin{bmatrix} -(-1)^t & \frac{5^t}{2} \\ (-1)^t & 5^t \end{bmatrix}\begin{bmatrix} -2/3 & 1/3 \\ 2/3 & 2/3 \end{bmatrix}$$

$$A^t = \frac{1}{3}\begin{bmatrix} (5)^t + 2(-1)^t & (5)^t - (-1)^t \\ 2(5)^t - 2(-1)^t & 2(5)^t + (-1)^t \end{bmatrix}$$

can check, let $t = 3$:

$$A^3 = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} = \begin{bmatrix} 9 & 8 \\ 16 & 17 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} = \begin{bmatrix} 41 & 42 \\ 84 & 83 \end{bmatrix}$$

$$A^3 = \frac{1}{3}\begin{bmatrix} (5)^3 + 2(-1)^3 & (5)^3 - (-1)^3 \\ 2(5)^3 - 2(-1)^3 & 2(5)^3 + (-1)^3 \end{bmatrix}$$

$$= \frac{1}{3}\begin{bmatrix} 125 - 2 & 125 + 1 \\ 250 + 2 & 250 - 1 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 123 & 126 \\ 252 & 249 \end{bmatrix} = \begin{bmatrix} 41 & 42 \\ 84 & 83 \end{bmatrix}$$

Example – Discrete Dynamical Systems

The system $\vec{x}(t+1) = A\vec{x}(t)$ is called a discrete dynamical system.

$\vec{x}(t)$ is called the state vector of the system at time t. "Discrete" indicates that we model the change of the system from time t to $t+1$.

Looking at each step, starting at time $t = 0$, we get

$$\begin{aligned} \vec{x}(1) &= A\vec{x}(0) \\ \vec{x}(2) &= A\vec{x}(1) = A(A\vec{x}(0)) = A^2\vec{x}(0) \\ \vec{x}(3) &= A\vec{x}(2) = A(A^2\vec{x}(0)) = A^3\vec{x}(0) \end{aligned}$$

$$\therefore \quad \vec{x}(t) = A^t\vec{x}(0)$$

Example Consider the dynamical system $\vec{x}(t) = A^t\ \vec{x}(0)$ where $A = \begin{bmatrix} 1/2 & 3/4 \\ 1/2 & 1/4 \end{bmatrix}$ and $\vec{x}(0) = \begin{bmatrix} 100 \\ 0 \end{bmatrix}$. Find a closed formula for $\vec{x}(t)$, and find $\lim_{t\to\infty} \vec{x}(t)$.

$$A = \begin{bmatrix} 1/2 & 3/4 \\ 1/2 & 1/4 \end{bmatrix} \qquad \text{Find a formula for } A^t$$

$$\det \begin{bmatrix} \lambda - 1/2 & -3/4 \\ -1/2 & \lambda - 1/4 \end{bmatrix} = 0 \quad \Rightarrow \quad \lambda^2 - \frac{3}{4}\lambda + \frac{1}{8} - \frac{3}{8} = \lambda^2 - \frac{3}{4}\lambda - \frac{1}{4}$$

$$= (\lambda - 1)(\lambda + \frac{1}{4}) = 0$$

$$\Rightarrow \quad \lambda = -1/4 \quad \text{or} \quad \lambda = 1$$

$$E_{-1/4} = \ker\left(\begin{bmatrix} -3/4 & -3/4 \\ -1/2 & -1/2 \end{bmatrix}\right) \quad = \quad \ker\left(\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\right) = t\begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

where t is arbitrary

$$E_1 = \ker\left(\begin{bmatrix} 1/2 & -3/4 \\ -1/2 & 3/4 \end{bmatrix}\right) \quad = \quad \ker\left(\begin{bmatrix} 1 & -3/2 \\ 0 & 0 \end{bmatrix}\right) = t\begin{bmatrix} 3/2 \\ 1 \end{bmatrix} = t\begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

where t is arbitrary

$$S = \begin{bmatrix} 3 & -1 \\ 2 & 1 \end{bmatrix} \quad \text{and} \quad S^{-1} = \frac{1}{5}\begin{bmatrix} 1 & 1 \\ -2 & 3 \end{bmatrix}$$

$$D = \begin{bmatrix} 1 & 0 \\ 0 & -\frac{1}{4} \end{bmatrix} \quad \text{and} \quad D^t = \begin{bmatrix} 1 & 0 \\ 0 & (-\frac{1}{4})^t \end{bmatrix}$$

$$A^t = SD^tS^{-1} = \frac{1}{5}\begin{bmatrix} 3 & -1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & (-\frac{1}{4})^t \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -2 & 3 \end{bmatrix}$$

$$A^t = \frac{1}{5}\begin{bmatrix} 3 & -(-\frac{1}{4})^t \\ 2 & (-\frac{1}{4})^t \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -2 & 3 \end{bmatrix}$$

$$A^t = \frac{1}{5}\begin{bmatrix} 3+2(-\frac{1}{4})^t & 3-3(-\frac{1}{4})^t \\ 2-2(-\frac{1}{4})^t & 2+3(-\frac{1}{4})^t \end{bmatrix}$$

we know that $\vec{x}(t) = A^t\,\vec{x}(0) = A^t \begin{bmatrix} 100 \\ 0 \end{bmatrix}$

$$\Rightarrow \quad \vec{x}(t) = \begin{bmatrix} 60+40(-\frac{1}{4})^t \\ 40-40(-\frac{1}{4})^t \end{bmatrix}$$

and $\lim_{t\to\infty} \vec{x}(t) = \begin{bmatrix} 60 \\ 40 \end{bmatrix}$

the equilibrium state of this system is $\begin{bmatrix} 60 \\ 40 \end{bmatrix}$