Consider the function $f$ from $\mathbb{R}$ to $\mathbb{R}$, or $y = f(x)$, where $x \in \mathbb{R}$ and $y \in \mathbb{R}$. Recall that the set in which the input variables reside (the $x$ variables in this case) is called the **domain** of $f$, and the set in which the output variables reside (the $y$ variables in this case) is called the **codomain** of $f$. So, for this example, the domain of $f$ is $\mathbb{R}$ and the codomain of $f$ is $\mathbb{R}$.

We also know that functions do not necessarily take on all values in their codomain. The set of values that the function takes in its codomain is called the **image** of the function.

For example, consider the function $f$ from $\mathbb{R}$ to $\mathbb{R}$ that is defined specifically as $y = f(x) = e^x$. The codomain of $f$ is $\mathbb{R}$, but we know that $e^x > 0$, thus the image of $f$ is all real numbers $> 0$.

If $f$ is from $X$ to $Y$, then 

$$\text{image}(f) = \{y \in Y : y = f(x), \text{ for some } x \in X\}$$

note that the image($f$) is a subset of the codomain of $f$.

Now consider the linear transformation $T$ from $\mathbb{R}^n$ to $\mathbb{R}^m$, or $\vec{y} = T(\vec{x}) = A\vec{x}$, where $\vec{x} \in \mathbb{R}^n$ and $\vec{y} \in \mathbb{R}^m$, and $A$ is an $m \times n$ matrix. The domain of $T$ is $\mathbb{R}^n$ and the codomain of $T$ is $\mathbb{R}^m$. But what is the **image** of $T$?
The image of $T$ is the set of values that the $\vec{y}$ can take. If $A$ is written as:

$$A = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \end{bmatrix}$$

then

$$\vec{y} = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1 \vec{v}_1 + x_2 \vec{v}_2 + \ldots + x_n \vec{v}_n$$

where $x_1, x_2, \ldots, x_n$ are arbitrary scalars and $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ are vectors in $\mathbb{R}^m$.

Therefore, the image of $T$ is all possible values of $\vec{y}$, or all possible values of $x_1 \vec{v}_1 + x_2 \vec{v}_2 + \ldots + x_n \vec{v}_n$, where $x_1, x_2, \ldots, x_n$ are arbitrary scalars, which is just the span of the set of vectors $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$.

$$\text{image}(T) = \text{span}(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n)$$

or

$$\text{im}(T) = \text{span of the columns of } A$$
Kernel of a Linear Transformation

We’re often interested in the zeros of a function; i.e., solutions of the equation \( f(x) = 0 \). The same is true of linear transformations.

The kernel of a linear transformation is the set of all zeros of the transformation, or the set of all \( \vec{x} \) such that \( T(\vec{x}) = A\vec{x} = \vec{0} \). The kernel of \( T \) is denoted by \( \ker(T) \).

\[ \begin{align*}
\therefore \quad \text{Given the linear transformation } T \text{ from } \mathbb{R}^n \text{ to } \mathbb{R}^m, \text{ where } T(\vec{x}) = A\vec{x}, \text{ the kernel of } T \text{ is the set of all } \vec{x} \in \mathbb{R}^n \text{ such that } A\vec{x} = \vec{0} \in \mathbb{R}^m.
\end{align*} \]

Or, the \( \ker(T) \) is the solution(s) to the linear system \( A\vec{x} = \vec{0} \).

Note the the \( \ker(T) \) is a subset of the domain of \( T \), just as the \( \text{im}(T) \) was a subset of the codomain of \( T \).

Since the right hand side of the system \( A\vec{x} = \vec{0} \) are all 0’s, then the system will always be consistent. Therefore, there is either a unique solution or an infinite number of solutions.

We also know that the \( \vec{0} \) vector is always in the \( \ker(T) \), therefore

\[ \text{unique solution to } A\vec{x} = \vec{0} \quad \Rightarrow \quad \ker(T) = \vec{0}. \]
Image and Kernel of an Invertible Linear Transformation

If the linear transformation $T$ from $\mathbb{R}^n$ to $\mathbb{R}^m$ is invertible, then there is a one-to-one correspondence between the $\vec{x}$ in the domain $\mathbb{R}^n$ and the $\vec{y}$ in the codomain $\mathbb{R}^m$.

\[ \therefore \text{ for each } \vec{y} \text{ in the codomain, there is one and only one } \vec{x} \text{ in the domain, namely } \vec{x} = A^{-1} \vec{y}. \]

\[ \Rightarrow \text{im}(T) = \text{codomain of } T. \]

Therefore, $T$ invertible $\Rightarrow$ im$(T) = \text{codomain of } T.$

The one-to-one correspondence also implies that there is only one vector $\vec{x}$ such that $A\vec{x} = \vec{0}$. As shown above, this vector must be the $\vec{0}$ vector.

Therefore, $T$ invertible $\Rightarrow$ ker$(T) = \vec{0}.$
Basis of the Image and Kernel of a Linear Transformation

Suppose we are given a linear transformation $T$ from $\mathbb{R}^n$ to $\mathbb{R}^m$. If we can find a set of vectors that forms a basis of $\text{im}(T)$, then we know that $\text{im}(T)$ is all possible linear combinations of those basis vectors. The number of vectors in the basis of $\text{im}(T)$, or $\dim(\text{im}(T))$, will give us a better picture of the image itself.

For example, suppose that $T$ is a linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^3$. If we find that the basis of $\text{im}(T)$ consists of one vector, then the image is simply a line in $\mathbb{R}^3$ (any scalar multiple of a single vector in $\mathbb{R}^3$). If we find that the basis of $\text{im}(T)$ consists of two vectors, then the image is simply a plane in $\mathbb{R}^3$ (any linear combination of two vectors in $\mathbb{R}^3$). If we find that the basis of $\text{im}(T)$ consists of three vectors, then the image is the entire codomain, or all of $\mathbb{R}^3$ (any linear combination of three vectors that form a basis of $\mathbb{R}^3$).

The same is true for the $\text{ker}(T)$. That is, if we can find a set of vectors that forms a basis of $\text{ker}(T)$, then we know that $\text{ker}(T)$ is all possible linear combinations of those basis vectors. The number of vectors in the basis of $\text{ker}(T)$, or $\dim(\text{ker}(T))$, will give us a better picture of the kernel itself.

For example, suppose that $T$ is a linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^3$. If we find that the basis of $\text{ker}(T)$ consists of one vector, then the kernel is simply a line in $\mathbb{R}^3$ (any scalar multiple of a single vector in $\mathbb{R}^3$). If we find that the basis of $\text{ker}(T)$ consists of two vectors, then the kernel is simply a plane in $\mathbb{R}^3$ (any linear combination of two vectors in $\mathbb{R}^3$). If we find that the basis of $\text{ker}(T)$ consists of three vectors, then the kernel is the entire domain, or all of $\mathbb{R}^3$ (any linear combination of three vectors that form a basis of $\mathbb{R}^3$).
Finding Bases for the Image and Kernel of a Linear Transformation

There is a systematic way to find a set of vectors that forms a basis of \( \text{im}(T) \) and a set of vectors that forms a basis of \( \text{ker}(T) \), where \( T \) is a linear transformation given by \( T(\vec{x}) = A\vec{x} \). In this method, we will interchange the idea of the image and kernel of a linear transformation with the idea of the image and kernel of a matrix; or, we assume that \( \text{im}(A) \equiv \text{im}(T) \) and \( \text{ker}(A) \equiv \text{ker}(T) \).

To find the bases for the image and kernel of a matrix \( A \):

1. Consider the system \( A\vec{x} = \vec{0} \).
2. Use elimination to reduce the system to rref.
3. Identify the leading 1’s.
4. The columns in \( A \) corresponding to the columns in rref(\( A \)) with leading 1’s form a basis of \( \text{im}(A) \).
5. If the solution to the system is unique (the vector \( \vec{0} \)), then \( \text{ker}(A) = \vec{0} \), there are no vectors in the basis of \( \text{ker}(A) \), and thus \( \text{dim}(\text{ker}(A)) = 0 \).
6. If there are an infinite number of solutions to the system (remember, either unique solution or infinite number of solutions when the right hand side is all 0’s), write the solution of the system as a linear combination of arbitrary variables and vectors. The vectors in the solution form a basis of \( \text{ker}(A) \).
Example:

Given the linear transformation \( T \) from \( \mathbb{R}^4 \) to \( \mathbb{R}^4 \) given by \( T(\vec{x}) = A\vec{x} \), where

\[
A = \begin{bmatrix}
1 & -4 & -2 & 3 \\
1 & -4 & 1 & 6 \\
-1 & 3 & 2 & -1 \\
1 & -1 & -2 & -3
\end{bmatrix}
\]

find a basis for the \( \text{im}(T) \) and a basis for the \( \ker(T) \).

First write the system \( A\vec{x} = \vec{0} \) in augmented matrix form and use elimination to reduce the system to rref:

\[
\begin{bmatrix}
1 & -4 & -2 & 3 & : & 0 \\
1 & -4 & 1 & 6 & : & 0 \\
-1 & 3 & 2 & -1 & : & 0 \\
1 & -1 & -2 & -3 & : & 0
\end{bmatrix} \rightarrow \begin{bmatrix}
1 & 0 & 0 & -3 & : & 0 \\
0 & 1 & 0 & -2 & : & 0 \\
0 & 0 & 1 & 1 & : & 0 \\
0 & 0 & 0 & 0 & : & 0
\end{bmatrix}
\]

\( \vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4 \) \( \rightarrow \) \( \vec{r}_1, \vec{r}_2, \vec{r}_3, \vec{r}_4 \)

The first three columns of \( \text{rref}(A) \) have leading 1’s \( \Rightarrow \) \{\( \vec{v}_1, \vec{v}_2, \vec{v}_3 \)\} forms a basis of \( \text{im}(A) \), or

\[
\left\{ \begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} -4 \\ -4 \\ 3 \\ -1 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 2 \\ -2 \end{bmatrix} \right\} = \text{basis of } \text{im}(A)
\]

\( \Rightarrow \) \( \text{im}(A) = \) all linear combinations of the basis vectors, or

\[
\text{im}(A) = c_1 \vec{v}_1 + c_2 \vec{v}_2 + c_3 \vec{v}_3, \text{ where } c_1, c_2, c_3 \text{ are arbitrary.}
\]
Now find the solution to the system:

Assume the fourth column corresponds to a variable $x_4$. This column does not have a leading 1, so it is nonleading. Therefore, let $x_4 = t$, where $t$ is arbitrary.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 3t \\ 2t \\ -t \\ t \end{bmatrix} = t \begin{bmatrix} 3 \\ 2 \\ -1 \\ 1 \end{bmatrix}, \quad t \text{ arbitrary}$$

$$\begin{bmatrix} 3 \\ 2 \\ -1 \\ 1 \end{bmatrix} \text{ forms a basis of } \ker(A)$$

$$\Rightarrow \ker(A) = \text{ all linear combinations of the basis vectors, or}$$

$$\ker(A) = c_1 \begin{bmatrix} 3 \\ 2 \\ -1 \\ 1 \end{bmatrix}, \text{ where } c_1 \text{ is arbitrary.}$$
Rank-Nullity Theorem

Notice that the number of vectors in the basis of the im(A) will always be equal to the number of leading 1’s, which is just the rank(A), or

$$\dim(\text{im}(A)) = \text{rank}(A)$$

Also notice that the number of vectors in the basis of the ker(A) will always be equal to the number of nonleading variables, which will always be the number of columns of A minus the number of leading 1’s, or

$$\dim(\text{ker}(A)) = n - \text{rank}(A)$$

The dimension of the ker(A) is defined as the nullity.

$$\therefore \ \dim(\text{im}(A)) + \dim(\text{ker}(A)) = \text{rank}(A) + \text{nullity}(A) = \text{rank}(A) + n - \text{rank}(A),$$

or

$$\text{rank}(A) + \text{nullity}(A) = n \quad (\text{Rank-Nullity Theorem}),$$

where $n$ is the number of columns in $A$. 