Linear Systems Involving Infinite Number of Solutions

Linear systems that have an infinite number of solutions will involve one or more “arbitrary” parameters. Arbitrary parameters can take on any value. We will want to express the solutions to these types of systems in terms of the least number of arbitrary parameters.

For linear systems of equations resulting in an infinite number of solutions, the procedure for expressing the solution in terms of the least number of arbitrary parameters is given below.

1. Use Gauss-Jordan elimination to reduce the system to reduced row-echelon form
2. Identify the leading ones
3. Identify columns with no leading ones
   — recall that variables corresponding to these columns are called “nonleading” variables
4. Assign an arbitrary parameter (r, s, t, etc.) to each nonleading variable
5. Using each equation (row), solve for all variable in terms of the arbitrary parameters
6. Write the solution vector as a sum of arbitrary parameters × vectors
Matrix-Matrix Multiplication

Given an $m \times n$ matrix $A$ and a $q \times p$ matrix $B$, how do we find the matrix-matrix product $AB$?, written

$$AB = [A] [B] = [C]$$

$$m \times n \quad q \times p$$

Recall from last lecture that the matrix-vector product $A\vec{x}$ was defined as:

$$A\vec{x} = \begin{bmatrix} | & | & | \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$$= x_1\vec{v}_1 + x_2\vec{v}_2 + \cdots + x_n\vec{v}_n \quad \text{(vector form)}$$

and exists only if the number of columns of $A$ is equal to the number of rows (components) of the vector $\vec{x}$.

Can think of the matrix-matrix product $AB$ as an extension of the matrix-vector product $A\vec{x}$ (where $\vec{x}$ was simply a 1-column matrix)

$$AB = A \begin{bmatrix} | & | & | \\ \vec{w}_1 & \vec{w}_2 & \cdots & \vec{w}_p \end{bmatrix} = \begin{bmatrix} | & | & | \\ \vec{w}_1 & \vec{w}_2 & \cdots & \vec{w}_p \end{bmatrix}$$

$$m \times n \quad q \times p \quad m \times p$$
In this case, each column of the matrix-matrix product is a matrix-vector product. This is called computing the product “column-by-column.”

The matrix-matrix product will exist only if the number of columns in the first matrix is equal the number of rows in the second matrix (or $n = q$ in our example).

Also note that the resulting product $AB$ is a $m \times p$ matrix, which is the same number of rows as $A$ and the same number of columns as $B$.

In summary, the matrix-matrix product, $C$, of $A$ and $B$ will exist only if the number of columns of $A$ equals the number of rows of $B$, and $C$ will have the same number of rows as $A$ and the same number of columns as $B$.

Now look at the matrix-matrix product in terms of its entries:

$$AB = \begin{bmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
a_{21} & a_{22} & \cdots & a_{2n} \\
\vdots & \vdots & & \vdots \\
a_{m1} & a_{m2} & \cdots & a_{mn}
\end{bmatrix} \begin{bmatrix}
b_{11} & b_{12} & \cdots & b_{1p} \\
b_{21} & b_{22} & \cdots & b_{2p} \\
\vdots & \vdots & & \vdots \\
b_{n1} & b_{n2} & \cdots & b_{np}
\end{bmatrix}
\quad m \times n \quad n \times p
$$

should give an $m \times p$ matrix

$$= \begin{bmatrix}
c_{11} & c_{12} & \cdots & c_{1p} \\
c_{21} & c_{22} & \cdots & c_{2p} \\
\vdots & \vdots & & \vdots \\
c_{m1} & c_{m2} & \cdots & c_{mp}
\end{bmatrix}
\quad m \times p$$
From the “column-by-column” approach, we know that the first column of $C$ is equal to the matrix-vector product $A\mathbf{w}_1$, where $\mathbf{w}_1$ is the first column of $B$. This says that:

$$
\begin{bmatrix}
c_{11} \\
c_{21} \\
\vdots \\
c_{m1}
\end{bmatrix} = A\mathbf{w}_1 = A
\begin{bmatrix}
b_{11} \\
b_{21} \\
\vdots \\
b_{n1}
\end{bmatrix} = b_11
\begin{bmatrix}
a_{11} \\
a_{21} \\
\vdots \\
a_{m1}
\end{bmatrix} + b_{21}
\begin{bmatrix}
a_{12} \\
a_{22} \\
\vdots \\
a_{m2}
\end{bmatrix} + \ldots + b_{n1}
\begin{bmatrix}
a_{1n} \\
a_{2n} \\
\vdots \\
a_{mn}
\end{bmatrix}
$$

or

$$
\begin{bmatrix}
c_{11} \\
c_{21} \\
\vdots \\
c_{m1}
\end{bmatrix} =
\begin{bmatrix}
a_{11}b_{11} + a_{12}b_{21} + \ldots + a_{1n}b_{n1} \\
a_{21}b_{11} + a_{22}b_{21} + \ldots + a_{2n}b_{n1} \\
\vdots \\
a_{m1}b_{11} + a_{m2}b_{21} + \ldots + a_{mn}b_{n1}
\end{bmatrix}
$$

Similarly for the second column of $C$:

$$
\begin{bmatrix}
c_{12} \\
c_{22} \\
\vdots \\
c_{m2}
\end{bmatrix} =
\begin{bmatrix}
a_{11}b_{12} + a_{12}b_{22} + \ldots + a_{1n}b_{n2} \\
a_{21}b_{12} + a_{22}b_{22} + \ldots + a_{2n}b_{n2} \\
\vdots \\
a_{m1}b_{12} + a_{m2}b_{22} + \ldots + a_{mn}b_{n2}
\end{bmatrix}
$$

\therefore for the $j$’th column of $C$:

$$
\begin{bmatrix}
c_{1j} \\
c_{2j} \\
\vdots \\
c_{mj}
\end{bmatrix} =
\begin{bmatrix}
a_{11}b_{1j} + a_{12}b_{2j} + \ldots + a_{1n}b_{nj} \\
a_{21}b_{1j} + a_{22}b_{2j} + \ldots + a_{2n}b_{nj} \\
\vdots \\
a_{m1}b_{1j} + a_{m2}b_{2j} + \ldots + a_{mn}b_{nj}
\end{bmatrix}
$$

Note that the $i$’th component of the $j$’th column of $C$ can be written as:

$$
c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \ldots + a_{in}b_{nj}
$$
This can be expressed in a more compact notation as:

\[ c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \ldots + a_{in}b_{nj} = \sum_{k=1}^{n} a_{ik}b_{kj} \]

This is also identical to the product of the \( i \)'th row of \( A \) and the \( j \)'th column of \( B \); i.e.,

\[
\begin{align*}
  c_{ij} &= \begin{bmatrix}
    a_{i1} & a_{i2} & \cdots & a_{in}
  \end{bmatrix}
  \begin{bmatrix}
    b_{1j} \\
    b_{2j} \\
    \vdots \\
    b_{nj}
  \end{bmatrix} \\
  &= [i \text{'th row of } A] [j \text{'th column of } B]
\end{align*}
\]

\[
\begin{align*}
  1 \times n & \quad n \times 1 = 1 \times 1
\end{align*}
\]

In summary, the \( ij \) entry of the matrix-matrix product is simply the product of the \( i \)'th row of the first matrix and the \( j \)'th column of the second matrix.

**Corollary:** since the matrix-matrix product depends on which matrix is written first and which is second, the product is not commutative, or \( AB \neq BA \) in general.

**Example** \[
\begin{align*}
  AB &= \begin{bmatrix}
    1 & 2 & 3 \\
    4 & 5 & 6 \\
    7 & 8 & 9
  \end{bmatrix}
  \begin{bmatrix}
    1 & 2 \\
    3 & 4 \\
    5 & 6
  \end{bmatrix}
  = \begin{bmatrix}
    22 & 28 \\
    49 & 64 \\
    76 & 100
  \end{bmatrix}
\end{align*}
\]

\[
\begin{align*}
  3 \times 3 & \quad 3 \times 2 \quad 3 \times 2
\end{align*}
\]
Matrix Algebra

Sums of matrices, $A$ and $B$ of the same size

\[
\begin{bmatrix}
a_{11} & \cdots & a_{1n} \\
\vdots & \ddots & \vdots \\
a_{m1} & \cdots & a_{mn}
\end{bmatrix} + \begin{bmatrix}
b_{11} & \cdots & b_{1n} \\
\vdots & \ddots & \vdots \\
b_{m1} & \cdots & b_{mn}
\end{bmatrix} = \begin{bmatrix}
a_{11} + b_{11} & \cdots & a_{1n} + b_{1n} \\
\vdots & \ddots & \vdots \\
a_{m1} + b_{m1} & \cdots & a_{mn} + b_{mn}
\end{bmatrix}
\]

Scalar multiples of matrices

\[
k \begin{bmatrix}
a_{11} & \cdots & a_{1n} \\
\vdots & \ddots & \vdots \\
a_{m1} & \cdots & a_{mn}
\end{bmatrix} = \begin{bmatrix}
ka_{11} & \cdots & ka_{1n} \\
\vdots & \ddots & \vdots \\
ka_{m1} & \cdots & ka_{mn}
\end{bmatrix}
\]

Matrix-matrix multiplication is associative

\[(AB)C = A(BC)\]

Matrix-matrix multiplication is distributive

\[A(C + D) = AC + AD\]

Matrix-matrix multiplication is not commutative

\[AB \neq BA \text{ in general}\]

If $k$ is a scalar

\[(kA)B = A(kB) = k(AB)\]

For an $m \times n$ matrix $A$

\[AI_n = I_m A = A\]