Basis

If a set of vectors \( \vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n \) in \( \mathbb{R}^m \) span \( \mathbb{R}^m \) and are linearly independent, then they form a basis of \( \mathbb{R}^m \).

\[ \therefore \text{the set of vectors } \vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n \text{ in } \mathbb{R}^m \text{ will form a basis of } \mathbb{R}^m \text{ if rank}(A) = m \text{ and rank}(A) = n, \text{ where } A \text{ is the } m \times n \text{ matrix} \]

\[
A = \begin{bmatrix}
\vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n
\end{bmatrix}
\]

The only way that rank\((A) = m \text{ and rank}(A) = n \) is when \( m = n \) and \( \text{rref}(A) = I_m = I_n \).

\[ \therefore \text{the set of } m \text{ vectors } \vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m \text{ in } \mathbb{R}^m \text{ will form a basis of } \mathbb{R}^m \text{ only if } \text{rref}(A) = I_m, \text{ where } A \text{ is the square } m \times m \text{ matrix} \]

\[
A = \begin{bmatrix}
\vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_m
\end{bmatrix}
\]
Claim: Given a basis of \( \mathbb{R}^m \), any vector \( \vec{v} \) in \( \mathbb{R}^m \) can be expressed uniquely as a linear combination of the \( m \) basis vectors.

Proof: Suppose the vectors \( \vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m \) form a basis of \( \mathbb{R}^m \), and consider a vector \( \vec{v} \) in \( \mathbb{R}^m \). Since the basis vectors span \( \mathbb{R}^m \), the vector \( \vec{v} \) can be written as a linear combination of the \( \vec{v}_i \). We have to demonstrate that this representation is unique. To do so, we consider two representations of \( \vec{v} \), namely,

\[
\vec{v} = c_1 \vec{v}_1 + c_2 \vec{v}_2 + \cdots + c_m \vec{v}_m = d_1 \vec{v}_1 + d_2 \vec{v}_2 + \cdots + d_m \vec{v}_m
\]

By subtraction, we find

\[
(c_1 - d_1)\vec{v}_1 + (c_2 - d_2)\vec{v}_2 + \cdots + (c_m - d_m)\vec{v}_m = \vec{0}
\]

which is a relation among the \( \vec{v}_i \). Since the \( \vec{v}_i \) are linearly independent, we have \( c_i - d_i = 0 \), or \( c_i = d_i \), for all \( i \): The two representations of \( \vec{v} \) are identical, as claimed.

The number of vectors in a basis of any space is called the dimension of that space. As shown above, there must be \( m \) vectors in a basis of \( \mathbb{R}^m \). Therefore, the dimension of \( \mathbb{R}^m \) will always be \( m \), written \( \dim(\mathbb{R}^m) = m \).

Consider the space \( \mathbb{R}^m \) with \( \dim(\mathbb{R}^m) = m \).

a. We can find at most \( m \) linearly independent vectors in \( \mathbb{R}^m \).

b. We need at least \( m \) vectors to span \( \mathbb{R}^m \).

c. If \( m \) vectors in \( \mathbb{R}^m \) are linearly independent, then they form a basis of \( \mathbb{R}^m \).

d. If \( m \) vectors span \( \mathbb{R}^m \), then they form a basis of \( \mathbb{R}^m \).
Coordinates

Consider a basis $\beta$ of a subspace $V$ of $\mathbb{R}^m$, consisting of vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ in $\mathbb{R}^m$. Any vector $\vec{x}$ in $V$ can be written uniquely as

$$\vec{x} = c_1 \vec{v}_1 + c_2 \vec{v}_2 + \cdots + c_n \vec{v}_n$$

The scalars $c_1, c_2, \ldots, c_n$ are called the $\beta$-coordinates of $\vec{x}$, and the vector

$$\begin{bmatrix}
  c_1 \\
  c_2 \\
  \vdots \\
  c_n 
\end{bmatrix}$$

is called the $\beta$-coordinate vector of $\vec{x}$, denoted by $[\vec{x}]_\beta$.

Note that

$$\vec{x} = S [\vec{x}]_\beta$$

where $S$ is the $m \times n$ matrix

$$S = \begin{bmatrix}
  \vert & \vert & \cdots & \vert \\
  \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \\
  \vert & \vert & \cdots & \vert
\end{bmatrix}$$
The standard basis of $\mathbb{R}^m$ will be defined as the vectors $\vec{e}_i$, where

$$\vec{e}_i = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad \leftarrow \text{i'th component}$$

The coordinate vector of any $\vec{x}$ in $\mathbb{R}^m$ with respect to the standard basis of $\mathbb{R}^m$ is just $\vec{x}$. This follows directly from the equation

$$\vec{x} = S \begin{bmatrix} \vec{x} \end{bmatrix}_\beta$$

where in the case of $\beta =$ standard basis, $S = I_m$. 
The vectors $\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_n$ are called orthonormal if they are all unit vectors and orthogonal to one another:

$$\vec{u}_i \cdot \vec{u}_j = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j \end{cases}$$

Properties of orthonormal vectors:

1. Orthonormal vectors are linearly independent
2. Orthonormal vectors $\vec{u}_1, \ldots, \vec{u}_n$ in $\mathbb{R}^n$ form a basis of $\mathbb{R}^n$ (called an orthonormal basis).

Consider an orthonormal basis $\vec{u}_1, \ldots, \vec{u}_n$ of $\mathbb{R}^n$. Then

$$\vec{x} = (\vec{u}_1 \cdot \vec{x}) \vec{u}_1 + \cdots + (\vec{u}_n \cdot \vec{x}) \vec{u}_n$$

for all $\vec{x}$ in $\mathbb{R}^n$.

Therefore, when dealing with an orthonormal basis, it is much easier to find the coordinates $c_i$ of any vector $\vec{x}$, since $c_i = \vec{u}_i \cdot \vec{x}$. 