Eigenvalues and Eigenvectors

It is useful in many applications to find nonzero vectors \( \vec{v} \), such that \( A\vec{v} \) is a scalar multiple of \( \vec{v} \), or \( A\vec{v} = \lambda \vec{v} \) for some scalar \( \lambda \).

Consider an \( n \times n \) matrix \( A \). A nonzero vector \( \vec{v} \) in \( \mathbb{R}^n \) is called an eigenvector of \( A \) if \( A\vec{v} \) is a scalar multiple of \( \vec{v} \); that is, if \( A\vec{v} = \lambda \vec{v} \) for some scalar \( \lambda \) (\( \lambda \) may be 0). The scalar \( \lambda \) is called the eigenvalue associated with the eigenvector \( \vec{v} \).

Finding the Eigenvalues of a Matrix

\( \lambda \) is an eigenvalue of \( A \) if there is a nonzero vector \( \vec{v} \) in \( \mathbb{R}^n \) such that

\[
A\vec{v} = \lambda \vec{v}
\]

or

\[
\lambda \vec{v} - A\vec{v} = 0
\]

or

\[
(\lambda I_n - A)\vec{v} = 0
\]

from the definition of \( \ker(A) \), the solution \( \vec{v} \) to the system above is \( \ker(\lambda I_n - A) \).

\[
\therefore \text{ the condition that } \vec{v} \neq \vec{0} \text{ is equivalent to the condition that } \ker(\lambda I_n - A) \neq \{\vec{0}\}.
\]
from what we know about the invertibility of a matrix, \( \ker(\lambda I_n - A) \neq \{\vec{0}\} \) iff \( \lambda I_n - A \) is not invertible.

\[
\therefore \text{ the condition that } \vec{v} \neq \vec{0} \text{ is equivalent to the condition that } \lambda I_n - A \text{ is not invertible.}
\]

from what we know about determinants, \( \lambda I_n - A \) is not invertible iff \( \det(\lambda I_n - A) = 0 \).

\[
\therefore \text{ the condition that } \vec{v} \neq \vec{0} \text{ is equivalent to the condition that } \det(\lambda I_n - A) = 0.
\]

\[
\therefore \lambda \text{ is an eigenvalue of } A \text{ if } \det(\lambda I_n - A) = 0.
\]

**Example**

Find all the eigenvalues of the matrix \( A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} \).

\[
\Rightarrow \text{ look for numbers } \lambda \text{ s.t. } \det(\lambda I_2 - A) = 0
\]

\[
\det(\lambda I_2 - A) = \det \left( \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} \right)
\]

\[
= \det \left( \begin{bmatrix} \lambda - 1 & -2 \\ -4 & \lambda - 3 \end{bmatrix} \right)
\]

\[
= (\lambda - 1)(\lambda - 3) - 8
\]
\[ \begin{align*}
= & \quad \lambda^2 - 4\lambda + 3 - 8 \\
= & \quad \lambda^2 - 4\lambda - 5 \\
= & \quad (\lambda - 5)(\lambda + 1)
\end{align*} \]

\[ \det(\lambda I_2 - A) = 0 \quad \Rightarrow \quad (\lambda - 5)(\lambda + 1) = 0 \]

\[ \Rightarrow \quad \lambda = -1 \quad \text{or} \quad \lambda = 5 \]

\[ \therefore \quad \lambda = -1 \quad \text{and} \quad \lambda = 5 \quad \text{are the eigenvalues of} \quad A. \]

**Example**

Find all the eigenvalues of a diagonal matrix.

let \( A = \begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix} \)

\[ \lambda I_n - A = \begin{bmatrix} \lambda - a_{11} & 0 & \cdots & 0 \\ 0 & \lambda - a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda - a_{nn} \end{bmatrix} \]

\[ \det(\lambda I_n - A) = (\lambda - a_{11})(\lambda - a_{22}) \cdots (\lambda - a_{nn}) \]

\[ \det(\lambda I_n - A) = 0 \quad \text{when} \quad \lambda = a_{11}, a_{22}, \cdots a_{nn} \]

\[ \therefore \quad \text{the eigenvalues of a diagonal matrix are its diagonal entries.} \]
Example

Find all the eigenvalues of a triangular matrix.

\[
A = \begin{bmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
0 & a_{22} & \cdots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & a_{nn}
\end{bmatrix}
\]

\[
\lambda I_n - A = \begin{bmatrix}
\lambda - a_{11} & -a_{12} & \cdots & -a_{1n} \\
0 & \lambda - a_{22} & \cdots & -a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \lambda - a_{nn}
\end{bmatrix}
\]

\[
\det(\lambda I_n - A) = (\lambda - a_{11})(\lambda - a_{22}) \cdots (\lambda - a_{nn})
\]

\[
\det(\lambda I_n - A) = 0 \text{ when } \lambda = a_{11}, a_{22}, \cdots, a_{nn}
\]

\[
\therefore \text{ the eigenvalues of a triangular matrix are its diagonal entries.}
\]

We can think of the eigenvalues of an \( n \times n \) matrix \( A \) as the roots of the function \( f_A(\lambda) = \det(\lambda I_n - A) \). \( f_A(\lambda) \) is called the characteristic polynomial of \( A \).

The characteristic polynomial of the \( n \times n \) matrix

\[
A = \begin{bmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{n1} & a_{n2} & \cdots & a_{nn}
\end{bmatrix}
\]

will be of the form

\[
f_A(\lambda) = \lambda^n - (a_{11} + a_{22} + \cdots + a_{nn})\lambda^{n-1} + \cdots + (-1)^n \det(A)
\]
the sum of diagonal entries of a matrix is called the \textbf{trace} of the matrix, denoted \( \text{tr}(A) \).

\[ \therefore \text{ for a general } n \times n \text{ matrix } A, \text{ the characteristic polynomial is} \]
\[ f_A(\lambda) = \lambda^n - \text{tr}(A)\lambda^{n-1} + \cdots + (-1)^n \text{det}(A) \]

We know that a polynomial of degree \( n \) has at most \( n \) zeros.
\[ \Rightarrow \text{ an } n \times n \text{ matrix has at most } n \text{ eigenvalues.} \]

We also know from the intermediate value theorem that if \( f_A(\lambda) \) is of odd degree, then there is at least one zero.
\[ \Rightarrow \text{ if } n \text{ is odd, then an } n \times n \text{ matrix has at least one eigenvalue.} \]

\underline{Algebraic Multiplicity}

If the characteristic polynomial is of the form
\[ f_A(\lambda) = (\lambda - \lambda_0)^k g(\lambda) \]
for some polynomial \( g(\lambda) \) with \( g(\lambda_0) \neq 0 \). Then we say that eigenvalue \( \lambda_0 \) has \textbf{algebraic multiplicity} \( k \).

Can think of the algebraic multiplicity of any eigenvalue as simply the number of times that eigenvalue is repeated in the set of all eigenvalues.
Finding the Eigenvectors of a Matrix

Recall that the eigenvectors of an $n \times n$ matrix $A$ are the vectors that satisfy the equation $A\vec{v} = \lambda \vec{v}$, or $(\lambda I_n - A)\vec{v} = 0$. This is equivalent to the kernel of the matrix $\lambda I_n - A$.

The kernel of the matrix $\lambda I_n - A$ is called the eigenspace associated with the eigenvalue $\lambda$, denoted by $E_\lambda$.

\[ E_\lambda = \ker(\lambda I_n - A) \]

Note that $E_\lambda$ consists of all the solutions $\vec{v}$ of the linear system $A\vec{v} = \lambda \vec{v}$. The procedure for finding the eigenvalues, $\lambda$, of a matrix was constructed so that there would be nonzero vectors $\vec{v}$ associated with each $\lambda$. Therefore, the eigenspace $E_\lambda$ will not be a unique solution and must have an infinite number of solutions, including the zero vector (solutions to kernel problems are always consistent, therefore either unique solution or infinite number of solutions).

Because the eigenspace $E_\lambda$ will have an infinite number of solutions, we will be able to express it as a linear combination of a set of vectors (sum of the arbitrary parameters times vectors). This set of vectors will form a basis of the eigenspace and will be the eigenvectors associated with the eigenvalue $\lambda$.

Example

Find the eigenvectors of the matrix $A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$.

$\Rightarrow$ from previous example, eigenvalues are $\lambda_1 = -1, \lambda_2 = 5$
$E_{-1}$ (eigenspace for $\lambda = -1$) = $\ker(-I_2 - A)$

$$E_{-1} = \ker\left(\begin{bmatrix} -2 & -2 \\ -4 & -4 \end{bmatrix}\right) = \ker\left(\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\right)$$

$$E_{-1} = t \begin{bmatrix} -1 \\ 1 \end{bmatrix} \text{ where } t \text{ is arbitrary}$$

$E_5$ (eigenspace for $\lambda = 5$) = $\ker(5I_2 - A)$

$$E_5 = \ker\left(\begin{bmatrix} 4 & -2 \\ -4 & 2 \end{bmatrix}\right) = \ker\left(\begin{bmatrix} 1 & -1/2 \\ 0 & 0 \end{bmatrix}\right)$$

$$E_5 = t \begin{bmatrix} 1/2 \\ 1 \end{bmatrix} \text{ where } t \text{ is arbitrary}$$

$\therefore$ the eigenvectors are $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1/2 \\ 1 \end{bmatrix}$.

(note that there are actually an infinite number of vectors that will satisfy the eigenvalue/eigenvector conditions; i.e., all scalar multiples of the vectors given above, for example $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} 1/2 \\ 2 \end{bmatrix}$ are also eigenvectors of $A$)

can check to see if these vectors are the correct eigenvectors of $A$:

$$\begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = -1 \begin{bmatrix} -1 \\ 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} 1/2 \\ 1 \end{bmatrix} = 5 \begin{bmatrix} 1/2 \\ 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 5/2 \\ 5 \end{bmatrix} = \begin{bmatrix} 5/2 \\ 5 \end{bmatrix}$$
we can represent the eigenspaces and eigenvectors graphically as:

\[
\begin{bmatrix}
-1 \\
1
\end{bmatrix}
\text{, } t \text{ arbitrary}
\quad \text{(eigenspace associated with } \lambda = 5) \\

\begin{bmatrix}
\frac{1}{2} \\
1
\end{bmatrix}
\quad \text{and}
\begin{bmatrix}
-1 \\
1
\end{bmatrix}
\text{, } t \text{ arbitrary}
\quad \text{(eigenspace associated with } \lambda = -1) \\
\]

**Geometric Multiplicity**

Consider an eigenvalue \( \lambda \) of a matrix \( A \). Then the dimension of the eigenspace \( E_\lambda = \ker(\lambda I_n - A) \) is called the **geometric multiplicity** of eigenvalue \( \lambda \).

\[ \therefore \text{ the geometric multiplicity of } \lambda \text{ is the nullity of } (\lambda I_n - A). \]
Example

Find the algebraic and geometric multiplicities for each of the eigenvalues of the matrix

\[
A = \begin{bmatrix}
1 & 1 & 1 \\
0 & 0 & 1 \\
0 & 0 & 1 \\
\end{bmatrix}.
\]

upper triangular matrix \( \Rightarrow \) eigenvalues are the diagonal entries 0, 1

\[
\lambda_1 = 0 \quad \text{(with algebraic multiplicity of 1)} \\
\lambda_2 = 1 \quad \text{(with algebraic multiplicity of 2)}
\]

\[
E_0 = \ker \left( \begin{bmatrix}
-1 & -1 & -1 \\
0 & 0 & -1 \\
0 & 0 & -1 \\
\end{bmatrix} \right) = \ker \left( \begin{bmatrix}
1 & 1 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0 \\
\end{bmatrix} \right)
\]

\[
E_0 = t \begin{bmatrix}
-1 \\
1 \\
0 \\
\end{bmatrix} \quad \text{where } t \text{ is arbitrary}
\]

dimension of \( E_1 = 1 \) \( \Rightarrow \) geometric multiplicity of \( \lambda_2 \) is 1

\[
E_1 = \ker \left( \begin{bmatrix}
0 & -1 & -1 \\
0 & 1 & -1 \\
0 & 0 & 0 \\
\end{bmatrix} \right) = \ker \left( \begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0 \\
\end{bmatrix} \right)
\]

\[
E_1 = t \begin{bmatrix}
1 \\
0 \\
0 \\
\end{bmatrix} \quad \text{where } t \text{ is arbitrary}
\]

dimension of \( E_0 = 1 \) \( \Rightarrow \) geometric multiplicity of \( \lambda_1 \) is 1
Note that the algebraic multiplicity is not necessarily equal to the geometric multiplicity for a given eigenvalue. In fact, it can be shown that for any eigenvalue $\lambda$ of a matrix $A$,

$$(\text{geometric multiplicity of } \lambda) \leq (\text{algebraic multiplicity of } \lambda)$$

**Example** - to illustrate the relationship between algebraic multiplicities and geometric multiplicities of an eigenvalue of a matrix (not a proof of above inequality)

$$A = \begin{bmatrix} 1 & \cdots & \cdots \\ 0 & 2 & \cdots \\ 0 & 0 & 4 & \cdots \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 & 4 \end{bmatrix}$$

$\lambda_1 = 1$, $\lambda_2 = 2$, $\lambda_3 = 4$

$\lambda_3 = 4$ has algebraic multiplicity of 3, but what about its geometric multiplicity?

$$E_4 = \ker \begin{bmatrix} 3 & \cdots & \cdots \\ 0 & 2 & \cdots \\ 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \ker \begin{bmatrix} 1 & \cdots & \cdots \\ 0 & 1 & \cdots \\ 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

there could be leading 1’s in columns 4 & 5, but not 3

rank will be between 2 and 4 (can’t be 5)

nullity = 5 - rank

nullity will be between 1 and 3

geom mult of ($\lambda_3 = 4$) is between 1 and 3

$\therefore$ geometric multiplicity of ($\lambda_3 = 4$) $\leq$ algebraic multiplicity of ($\lambda_3 = 4$)