The answer to a 200 year old mathematical question.

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The Question

Besides the plane, catenoid and helicoid are there any other minimal surfaces that are complete, embedded and have finite topology (finite total curvature?).

Timeline: Page 1

- Plane
- Catenoid (Euler 1740)
- Helicoid (Meusnier 1776)
- Plateaus Problem (J. Plateau 1800s)
  Does a minimal surface exist given a boundary?
- Scherks minimal surfaces (1830s)
  First minimal surfaces found since Helicoid.

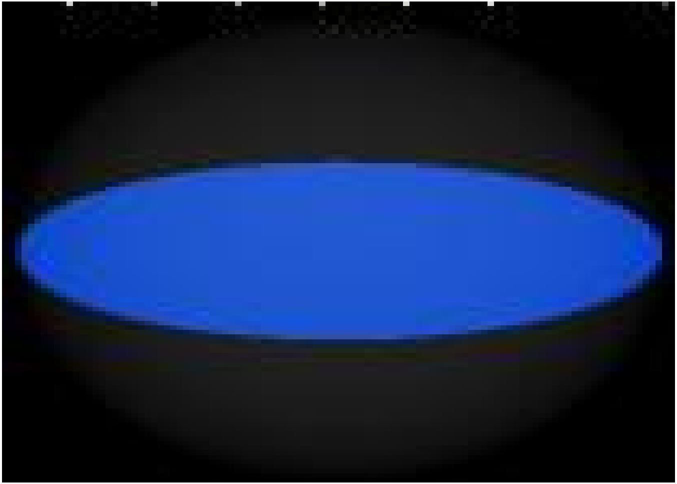
Timeline: Page 2

- Enneper-Weierstrass Formula (1863)
- Existence of Plateau Problem solutions (Douglas 1930s)
- Costas Minimal Surface (C. Costa 1982)
  A complete genus one minimal surface with three ends and finite topology.
- Graphics of Costas Surface (D. Hoffman, W. Meeks, J. Hoffman 1985).
  Symmetries and embedded.

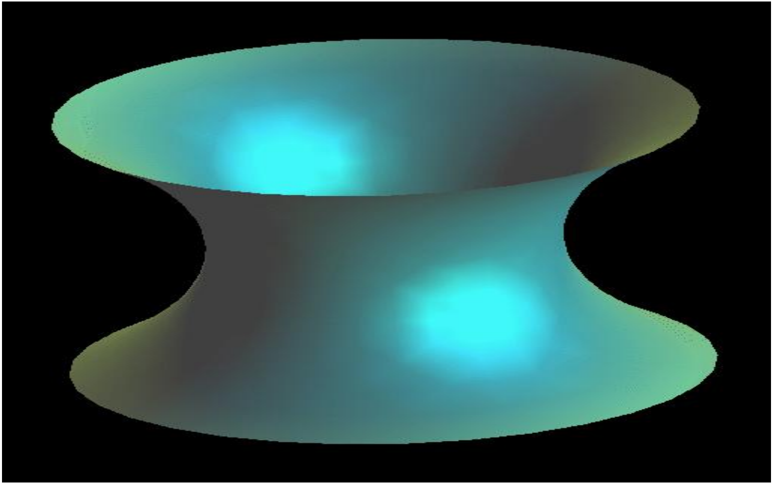
Timeline: Page 3

- Costa-Hoffman-Meeks (CHM) Surfaces (D. Hoffman, W. Meeks, J. Hoffman 1988).
  Family of minimal surfaces are complete embedded of finite topology.
- Explicit parametric representation (A. Gray 1996)
  Mathematica is used to visualize CHM minimal surfaces.
- Current work (1990's - today)
  Many other minimal surfaces and surfaces of constant mean curvature are created by using the tools developed by Hoffman, Meeks, Gray, Osserman, Bers, and others.

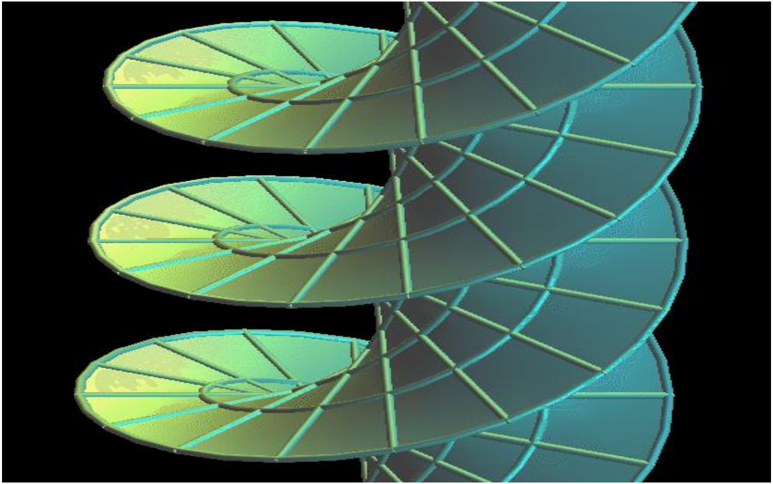
The Plane



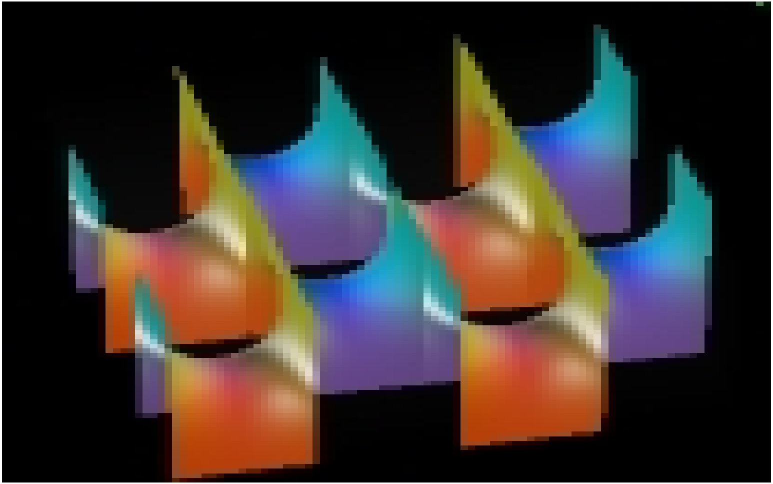
The Catenoid



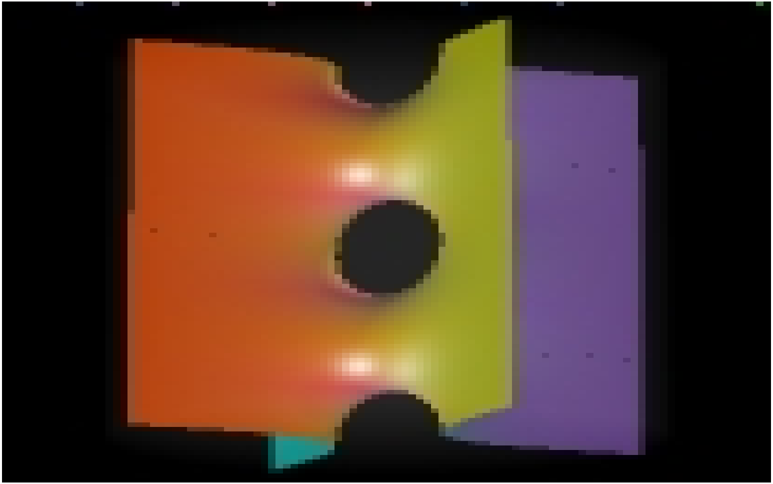
The Helicoid



Scherks First Surface



Scherks Second Surface



MS Definition - Neighborhood

Each point on the surface has a neighborhood which is the surface of least area with respect to its boundary.

MS Definition - Experimentalist

A sufficiently small piece of the surface may be modeled by a soap film spanning a contour which coincides with the boundary of the piece.

MS Definition - Meusniers

- The mean curvature H is defined as $H = \frac{1}{2}\{k_{max} + k_{min}\}$, where k_{max} and k_{min} are the maximum and minimum principle curvatures.
- The mean curvature H of a minimal surface vanishes identically, $H = 0$

MS Definition - Calculus of Variations

Euler-Lagrange equation for the area functional of a graph:

- If a surface is locally written as a graph of a real-valued function $z = f(x, y)$ then f satisfies the following nonlinear PDE:

$$f_{xx}(1 + f_y^2) - 2f_{xy}f_xf_y + f_{yy}(1 + f_x^2) = 0$$

The Question

Besides the plane, catenoid and helicoid are there any other minimal surfaces that are complete, embedded and have finite topology (finite total curvature?).

Complete Minimal Surface

A Complete Minimal Surface is a minimal surface without boundary.

Embedded Minimal Surface

An Embedded Minimal Surface is any minimal surface that is free of self-intersections.

A Surface with Finite Topology

- A surface is of finite topology if it is homeomorphic to a compact surface of genus n with m points removed.
- The Gaussian curvature, $K = k_{max}\, k_{min}$.
- The total curvature, C, is given by $C = \int_S KdA$.
- Theorem: The total curvature of a complete minimal surface with finite total curvature is given by

$$C = 2\pi(2 - 2n - 2m)$$

where n is the genus of the surface and m is the number of ends.

Examples of Finite Topology/Finite Total Curvature

Surface	Topology	Total Curvature
Plane	sphere with 1 end	0
Catenoid	sphere with 2 ends	-4π
Helicoid	sphere with 1 end	$-\infty$
Costa	torus with 3 ends	-12π

Note:

- The sphere has a topological genus $= 0$,
- the torus has topological genus $= 1$.

Enneper-Weierstrass(EW) Representation Formula

Given the two functions f and g, the function defined by

$$X(z) = \Re \int_{z_o}^{z} \Phi dz,$$

where

$$\Phi = (\phi_1, \phi_2, \phi_3) = [(1-g^2)f, i(1+g^2)f, 2fg]$$

defines a regular conformal minimal immersion.

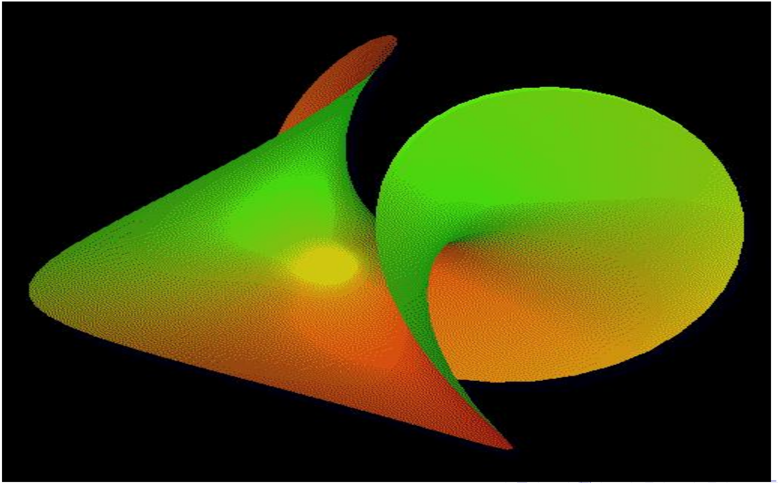
Examples Minimal Surfaces using the EW Representation Formula

Surface	Domain	f	g
Plane	$\mathbb{C}$	1	0
Enneper's surface	$\mathbb{C}$	1	z
Catenoid	$\mathbb{C} - \{0\}$	1	$\frac{1}{z}$
Bour's surface	$\mathbb{C}$	1	$\sqrt{2}$
Helicoid	$\mathbb{C} - \{0\}$	$\frac{dz}{z^2}$	z
Scherk's 1st Surface	$\hat{\mathbb{C}} - \{\lambda : \lambda^4 = 1\}$	$\frac{dz}{z^4-1}$	z
Scherk's 2st Surface	$\hat{\mathbb{C}} - \{\lambda : \lambda^4 = 1\}$	$\frac{idz}{z^4-1}$	z

Note : $\hat{\mathbb{C}}$ denotes $\mathbb{C} \cup \infty$.

Bour's Minimal Surface

Enneper's Minimal Surface



Costa's Theorem

The conformal minimal immersion $X : D \to \mathbb{R}^3$ defined by the Enneper- Weierstrass representation formula with

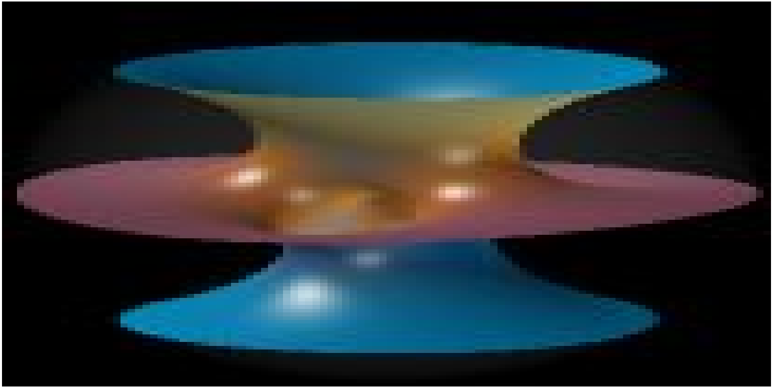
- D: $\mathbb{T}^2 - \{0, \frac{1}{2}, \frac{i}{2}\}$,
- $f = \wp$,
- $g = \frac{2\sqrt{2\pi}\wp(\frac{1}{2})}{\wp\prime}$,

where

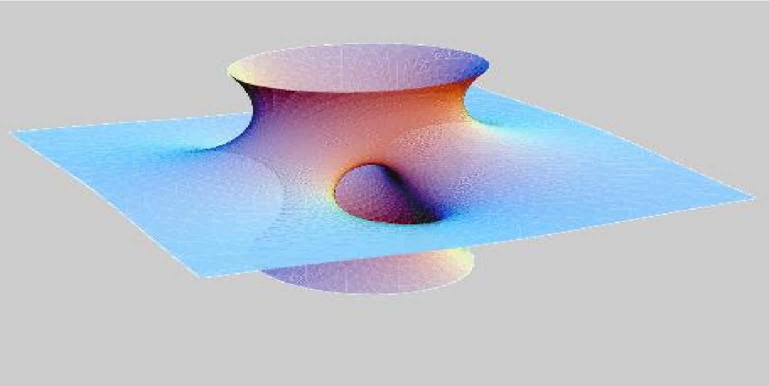
- $\mathbb{T}^2 = \mathbb{C}/L$ is the square torus,
- $L = \{n + im | n, m \in \mathbb{Z}\} \subset \mathbb{C}$,

defines a surface $M = X(D) \subset \mathbb{R}^3$ that is a complete minimal surface with total curvature $C_M = -12\pi$.

Costa's Minimal Surface - by Jim Hoffmann using MESH



Costa's Minimal Surface - by A. Gray using Mathematica



3D-XplorMath Costa Surface

Explicit Parametric Representation of Costa's Surface

- $x = \frac{1}{2}\Re\{-\zeta(\diamond) + \pi u + \frac{\pi^2}{4e_1} + \frac{\pi}{2e_1}[\zeta(\diamond - \frac{1}{2}) - \zeta(\diamond - \frac{i}{2})]\}$
- $y = \frac{1}{2}\Re\{-i\zeta(\diamond) + \pi v + \frac{\pi^2}{4e_1} - \frac{\pi}{2e_1}[i\zeta(\diamond - \frac{1}{2}) - i\zeta(\diamond - \frac{i}{2})]\}$
- $z = \frac{1}{4}\sqrt{2\pi}\ln\left|\frac{(\wp(\diamond)-e_1)}{(\wp(\diamond)+e_1)}\right|$

where

- $\diamond = u + iv$,
- $\zeta(\hat{z})$ is the Weierstrass zeta function,
- $\wp(g_2, g_3; z)$ is the Weierstrass elliptic function,
- $(g_2, g_3) = (189.072772..., 0)$ are the invariants corresponding to the half-periods $1/2$ and $i/2$,
- $e_1 \approx 6.87519$ is the first root.

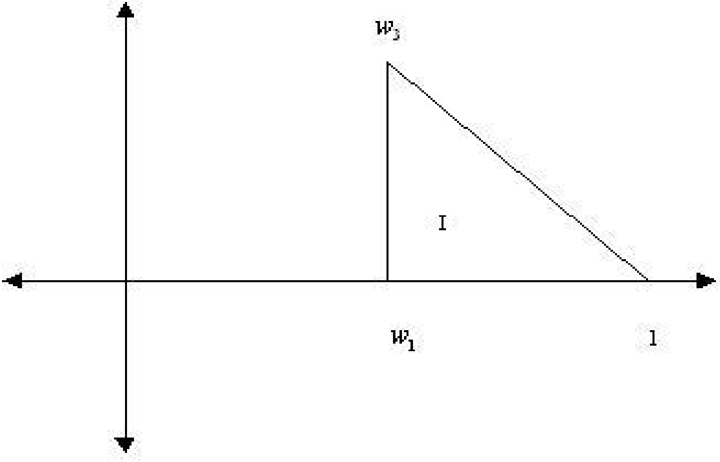
The Symmetries of Costa's Surface (Hoffman & Meeks)

The symmetries of Costa's Surface are the result of the symmetries of the function $\wp$. To construct the Weierstrass $\wp$ function:

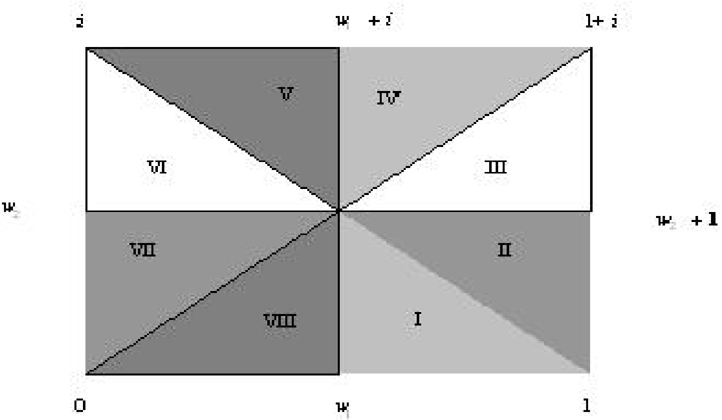
- Use a Schwarz-Chistoffel transformation t to map the upper half complex plane into a triangular domain I in the positive quadrant such that:
  - $t(0) = \omega_3$,
  - $t(e_1) = \omega_1$,
  - $t(\infty) = 1$.
- Define a function f such that $f(z) = \sqrt{t^{-1}(z)}$ with
  - $f(\omega_1) = e_1$,
  - $f(\omega_3) = 0$,
  - $f(1) = \infty$.

  where $e_1 \approx 6.875185$
- Construct the rest of the Weierstrass $\wp$ function by using Schwarz refection to extend the function f.

The triangle I



The Square F

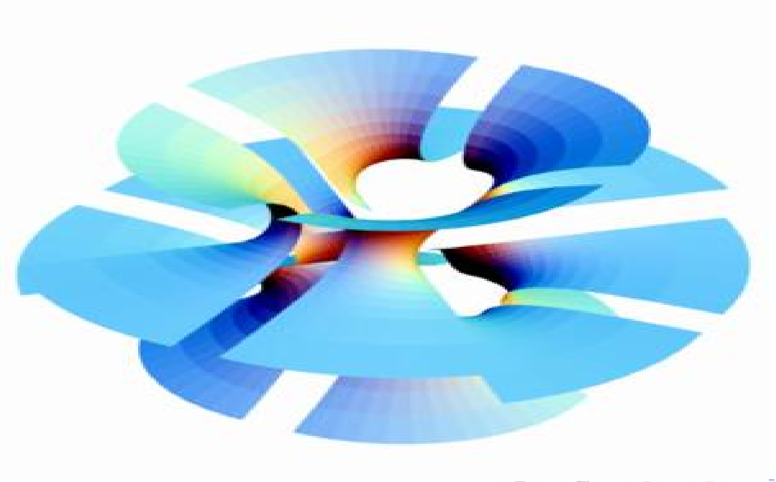


The Dihedral group D_4

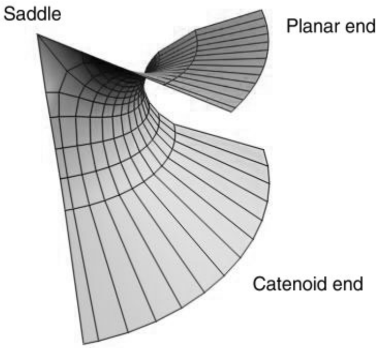
Theorem: (Hoffman and Meeks)

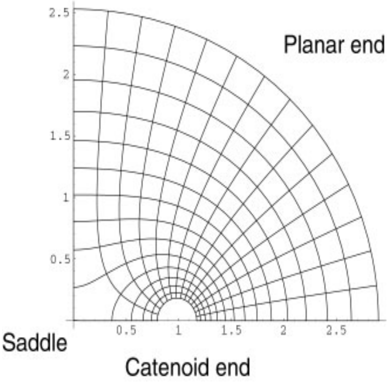
- The Dihedral group D_4 with 8 elements acts on the square F by reflections through the horizontal, vertical and diagonal lines through the point $\omega_3 = \frac{1}{2} + \frac{i}{2}$ and by rotations of integer multiples of $\frac{\pi}{2}$ about ω_3.

The Costa Surface Broken into 8 pieces

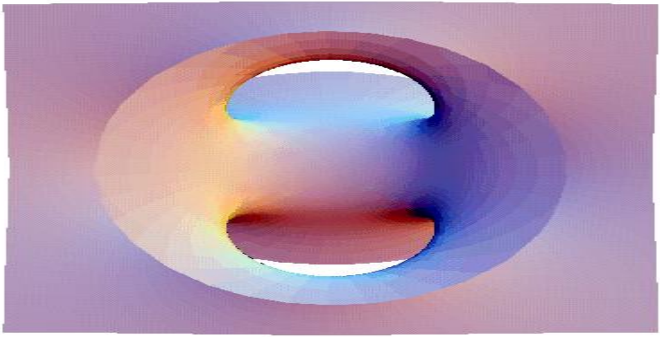


Parametrization of a singe piece

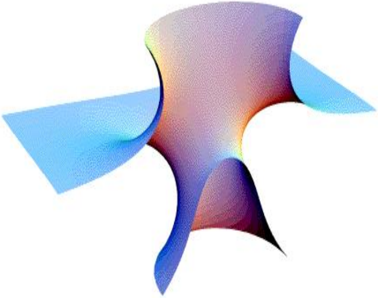


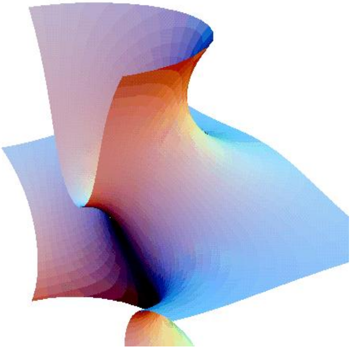


Top/Botton View

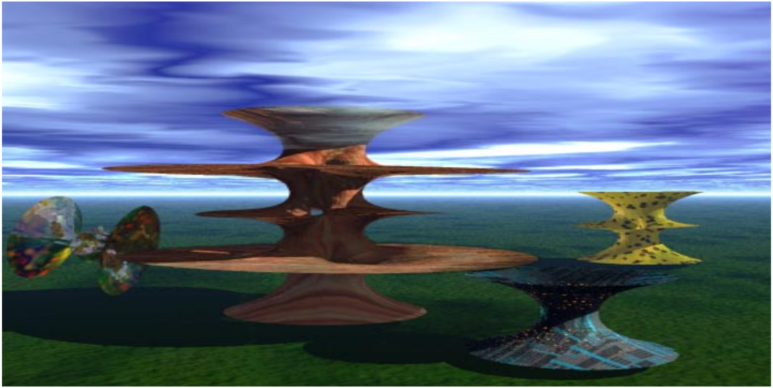


The Surface Cut





CHM Family of Minimal Surfaces



XplorMath CHM picture

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- Hoffman D. and Meeks W., *A Complete Minimal Surface in $\mathbb{R}^3$ with Genus One and Three Ends*, J. Differential Geometry 21 (1985), 109–127.
- Ferguson H., Gray A. and Markvorsen S., *Costa's Minimal Surface via Mathematica*, Mathematica in Education and Research, Vol. 5 No. 1 (1996), 5-10. TELOS/Springer-Verlag Publishers.

Web-sites Page 1

- GANG The Center for Geometry, Analysis, Numerics & Graphics, Dept of Mathematics & Statistics at the University of Massachusetts, Amherst, Massachusetts.

  http://www.gang.umass.edu/
- The Scientific Graphics Project (SGP) at The Mathematical Sciences Research Institute(MSRI), Berkeley, CA.

  http://www.msri.org/about/sgp/SGP/index.html
- 3D-XplorMath Gallery.

  http://vmm.math.uci.edu/3D-XplorMath/Surface/gallery.html
- Helaman Ferguson, Mathematician and Artist.

  www.helasculpt.com

Web-sites Page 2

- Stewart Dickson, Artist, Engineer and Computer Scientist.

  http://emsh.calarts.edu/ mathart/

- MathWorld

  http://mathworld.wolfram.com/CostaMinimalSurface.html

- Professor Alfred Gray Memorial Site

  http://math.cl.uh.edu/ gray/

- Matthias Weber

  http://www.indiana.edu/ minimal/toc.html

- Paul Nylander

  http://www.bugman123.com/index.html