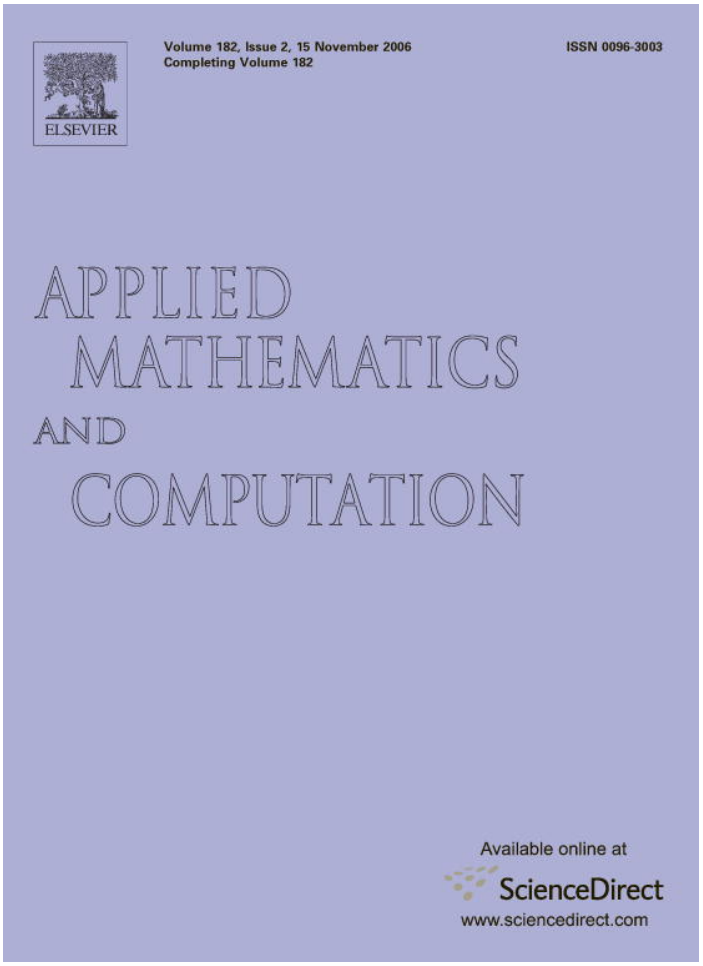

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A discontinuous Galerkin Method for parabolic problems with modified *hp*-finite element approximation technique

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Abstract

A recent paper [Hideaki Kaneko, Kim S. Bey, Gene J.W. Hou, Discontinuous Galerkin finite element method for parabolic problems, preprint November 2000, NASA] is generalized to a case where the spatial region is taken in R^3. The region is assumed to be a thin body, such as a panel on the wing or fuselage of an aerospace vehicle. The traditional *h*- as well as *hp*-finite element methods are applied to the surface defined in the *x*–*y* variables, while, through the thickness, the technique of the *p*-element is employed. Time and spatial discretization scheme developed in Kaneko et al. (2000), based upon an assumption of certain weak singularity of $\|u_t\|_2$, is used to derive an optimal a priori error estimate for the current method.

Keywords: Discontinuous Galerkin method; Parabolic equations; Modified *hp*-finite element method

1. Introduction

In this paper, the discontinuous Galerkin method is applied to the following standard model problem of parabolic type:

Find u such that

$$\begin{aligned} &u_t(x,t) - \Delta u(x,t) = f(x,t), && x \in \Omega,\ t \in R^+, \\ &u(x,t) = 0, && x \in \partial\Omega,\ t \in R^+, \\ &u(x,0) = u_0(x), && x \in \Omega, \end{aligned} \tag{1.1}$$

where Ω is a closed and bounded set in R^3 with boundary $\partial\Omega$, $R^+ = (0,\infty)$, $\Delta u = \partial^2 u/\partial x^2 + \partial^2/\partial y^2 + \partial^2 u/\partial z^2$, $u_t = \partial u/\partial t$, and the functions f and u_0 are given data.

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The discontinuous Galerkin method is a robust finite element method that can deliver high-order numerical approximation using unstructured grids. In this paper, region Ω is assumed to be a thin body in R^3, such as a panel on the wing or fuselage of an aerospace vehicle. The traditional *h*- as well as *hp*-finite element approximations are used in the *x*–*y* variables, whereas, the *p*-finite element method developed, e.g., in [5,15], is used in the *z* variable which describes the region through the thickness. The application of the *p*-finite element method through the thickness of thin structure, as compared to applying the *h*- or *hp*-finite element discretization to all coordinate directions, enables us to avoid structuring elements in R^3 that are too thin to satisfy the required quasi-uniformity condition (e.g., see [7]) that is necessary to deliver stable numerical approximation. It should be emphasized that the current technique is different from the technique commonly known as hierarchical finite element approximation for plate problems. In the hierarchical finite element technique, a class of basis functions are chosen that depend on the differential equation and subsequently numerical solution is obtained by solving a hierarchy of two-dimensional problems. We are coining the term 'modified *hp*'-finite element method, as it differs from the traditional *hp*-finite element method which uses *h*- and *p*-finite elements on the same domain where the *h*-finite element method provides a refinement of the region and the *p*-finite element provides an enrichment. In Section 2, approximation power of the modified *hp*-finite element method will be investigated. In Section 3, the discontinuous Galerkin method with the modified *hp*-finite element approximation technique is established. Discontinuity is in time variable and time discretization is based upon the degree of singularity of $\|u_t\|_2$. The traditional *h*-finite element method is employed in time. The convergence analysis given in [9] will be used. The reader is also reminded of recently published important paper [16] by Schötzau and Schwab in which various time discretization techniques are discussed. For instance, an exponential convergence rate in time of *p*-finite element method is obtained in [16] despite the presence of singularity in the transient phase of the solution. Time discretization used there is geometric. Schötzau and Schwab's result extends the results in [3,4] in which no exponential convergence rate is reported. Also they discuss the *h*-finite element technique in time using a class of radical mesh and obtain the algebraic convergence rate which is optimal. The radical mesh was chosen by analyzing the incompatibility between initial and boundary data. The present authors [1] established a similar time discretization technique for the discontinuous Galerkin finite element method, *h*-version in time, which was based upon the singularity of $\|u_t\|_2$. Using this analysis, it is shown in [1] that the optimal algebraic convergence rate in time of the discontinuous Galerkin method can be obtained under more dispersed, therefore more computationally stable, radical mesh than the mesh used in [16]. Time discretization was extended to nonlinear problem in [14].

2. Approximation power of modified *hp* elements

Let $\omega \subseteq R^2$ and $\Gamma \subseteq R$ be convex regions. For simplicity, it is assumed that $\Gamma \equiv \left[-\frac{d}{2}, \frac{d}{2}\right]$ where $d = |\Gamma|$. For simplicity, the thickness, d, is assumed constant over the domain. The Sobolev space of order k defined on $\omega \times \Gamma$ is denoted by $H^k(\omega \times \Gamma)$ with the norm

$$\|u\|^2_{k,\omega\times\Gamma} = \sum_{0\leqslant|\alpha|\leqslant k} \|D^\alpha u\|^2_2,$$

where for each multi-integer $\alpha = (\alpha_1, \alpha_2, \alpha_3)$, we have let $|\alpha| = \alpha_1 + \alpha_2 + \alpha_3$ and

$$D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \partial x_3^{\alpha_3}}.$$

We note that the Sobolev norm reduces to the usual L_2 norm when $k = 0$. In this section, a best possible error estimate is derived for approximating an element in $H^k(\omega \times \Gamma)$ by the finite element function spaces. Let $K_{\xi,\eta}$ denote the master triangular element defined by

$$K_{\xi,\eta} = \{(\xi,\eta) \in R^2 : 0 \leqslant \eta \leqslant (1+\xi)\sqrt{3} \;\; -1 \leqslant \xi \leqslant 0 \text{ or } 0 \leqslant \eta \leqslant (1+\xi)\sqrt{3} \;\; 0 \leqslant \xi \leqslant 1\}.$$

Let $S^p(K_{\xi,\eta})$ denote the space of polynomials of degree $\leqslant p$ on $K_{\xi,\eta}$, i.e.,

$$S^p(K_{\xi,\eta}) = \text{span}\{\xi^i \eta^j : i,\ j = 0, 1, \ldots, p;\ i + j \leqslant p\}.$$

First, the shape functions for the master element $K_{\xi,\eta}$ are formed. To accomplish this, the barycentric coordinates are introduced via

$$\lambda_1 = \left(1 - \xi - \eta/\sqrt{3}\right)/2, \quad \lambda_2 = \left(1 + \xi - \eta/\sqrt{3}\right)/2, \quad \lambda_3 = \eta/\sqrt{3}. \tag{2.1}$$

λ_i's form a partition of unity and λ_i is identically equal to one at a vertex of $K_{\xi,\eta}$ and vanishes on the opposite side of $K_{\xi,\eta}$. The hierarchical shape functions on $K_{\xi,\eta}$ consists of internal as well as external functions. The normalized antiderivatives of the Legendre polynomials are defined by

$$\bar{\psi}_i(\zeta) = \sqrt{\frac{2i+1}{2}} \int_{-1}^{\zeta} P_i(t)\mathrm{d}t, \quad i = 1, 2, 3, \ldots \tag{2.2}$$

Now, the external shape functions consist of three nodal shape functions

$$N_i(\xi, \eta) = \lambda_i, \quad i = 1, 2, 3, \tag{2.3}$$

and $3(p-1)$ side shape functions $N_i^{[j]}(\xi,\eta), i = 1, \ldots, p-1, j = 1, 2, 3$. The index j indicates one of three sides of $k_{\xi,\eta}$. Noting that $\bar{\psi}_i(\pm 1) = 0$,

$$\bar{\psi}_i(\eta) = \frac{1}{4}(1 - \eta^2)\varphi_i(\eta), \quad i = 1, 2, 3, \ldots, \tag{2.4}$$

where $\varphi_i(\eta)$ is a polynomial of degree $i-1$. For instance, $\varphi_1(\eta) = -\sqrt{6}$, $\varphi_2(\eta) = -\sqrt{10}\eta$ and $\varphi_3(\eta) = \frac{\sqrt{14}}{4}(1 - 5\eta^2)$, etc. The side shape functions are constructed as follows:

$$\begin{aligned} N_i^{[1]}(\xi,\eta) &= \lambda_2\lambda_3\varphi_i(\lambda_3 - \lambda_2), \\ N_i^{[2]}(\xi,\eta) &= \lambda_3\lambda_1\varphi_i(\lambda_1 - \lambda_3), \quad i = 1, \ldots, p-1. \\ N_i^{[3]}(\xi,\eta) &= \lambda_1\lambda_2\varphi_i(\lambda_2 - \lambda_1), \end{aligned} \tag{2.5}$$

From (2.4) and (2.5), there are $3 + 3(p-1)$ shape functions. As $dim(S^p(K_{\xi,\eta})) = \frac{(p+1)(p+2)}{2}$, the remaining $\frac{(p-1)(p-2)}{2}$ basis elements are constructed in terms of internal shape functions. Clearly, nontrivial internal shape functions on $K_{\xi,\eta}$ exists only if $p \geqslant 3$. For $p = 3$, the bubble function on $K_{\xi,\eta}$ below serves as the internal function;

$$b_{K_{\xi,\eta}}(\xi,\eta) = \lambda_1\lambda_2\lambda_3 = \frac{\eta}{4\sqrt{3}}\left(\left(1 - \frac{\eta}{\sqrt{3}}\right)^2 - \xi^2\right). \tag{2.6}$$

Moreover, the collection $I^p(K_{\xi,\eta})$ of higher-order internal shape functions can be constructed from

$$I^p(K_{\xi,\eta}) = \{b_{K_{\xi,\eta}} v : v \in S^{p-3}(K_{\xi,\eta})\} = \{b_{K_{\xi,\eta}}\} \otimes S^{p-3}(K_{\xi\eta}), \quad p \geqslant 3.$$

Let T_h, $h > 0$, be a triangulation of ω. Let $x = Q_x^l(L_1, L_2, L_3)$ and $y = Q_y^l(L_1, L_2, L_3)$ be the element mappings of the standard triangle $K_{\xi,\eta}$ to the lth triangular element $K^l \in T_h$, e.g., the linear mapping onto K^l with vertices $\left\{\left(x_i^l, y_i^l\right)\right\}_{i=1}^3$,

$$Q_x^l(L_1, L_2, L_3) = \sum_{i=1}^{3} x_i^l L_i, \quad Q_y^l(L_1, L_2, L_3) = \sum_{i=1}^{3} y_i^l L_i.$$

The space of all polynomials of degree $\leqslant p$ on K^l is denoted by $S^p(K^l)$ and its basis can be formed from the shape functions of $S^p(K_{\xi,\eta})$ described above by transforming them under Q_x^l and Q_y^l. The finite element space $S^{p,l}(\omega, T_h)$ is now defined. For $\omega, p \geqslant 0$ and $k \geqslant 0$,

$$S^{p,k}(\omega, T_h) = \{u \in H^k(\omega : u|_K \in S^p(K), \ K \in T_h)\}. \tag{2.7}$$

Assume that a triangulation $\{T_h\}$, $h > 0$, of ω consists of $\{K_h^l\}_{l=1}^{M(h)}$ and that $h_l = \mathrm{diam}(K_h^l)$, for $l = 1, \ldots, M(h)$.

In the z-variable for through the thickness approximation, the local variable τ is defined in the reference element $[-1, 1]$ and Γ is mapped onto the reference element by Q_z, i.e.,

$$\Gamma = Q_z([-1, 1]), \quad z = Q_z(\tau).$$

Clearly, Q_z is a linear function defined by

$$z = Q_z(\tau) = \frac{1}{2}(1-\tau)\left(-\frac{d}{2}\right) + \frac{1}{2}(1+\tau)\frac{d}{2}, \quad \tau \in [-1,1].$$

The Jacobian of Q_z is constant

$$\frac{\mathrm{d}z}{\mathrm{d}\tau} = \frac{d}{2}.$$

In this paper, the basis functions of $P_p([-1,1])$ are taken to be the one-dimensional hierarchical shape functions. See [15] for a complete discussion of the basis elements used in the p and hp-finite element methods.

For example, in approximating an element in $H^l[-1,1]$, with $l=0$, $\psi_i(\tau) = P_{i-1}(\tau)$, $1 \leqslant i \leqslant p+1$, where P_{i-1} is the Legendre polynomial of degree $i-1$, form the hierarchical basis functions. With $l=1$, the external (ψ_1 and ψ_2) and internal (ψ_i, $i \geqslant 3$) shape functions are defined by

$$\begin{aligned} &\psi_1(\tau) = \frac{1-\tau}{2}, \quad \psi_2(\tau) = \frac{1+\tau}{2}, \\ &\psi_i(\tau) = \left(\frac{2i-3}{2}\right)^{1/2} \int_{-1}^{\tau} P_{i-2}(t)\mathrm{d}t, \quad 3 \leqslant i \leqslant p+1. \end{aligned} \tag{2.8}$$

Note that ψ_i's form an orthogonal family with respect to the energy inner product $(\cdot,\cdot)_E$,

$$(\psi_i, \psi_j)_E \equiv \int_{-1}^{1} \psi_i'(t)\psi_j'(t)\mathrm{d}t = \int_{-1}^{1} P_i(t)P_j(t)\mathrm{d}t = \delta_{ij}.$$

Also note that the internal shape functions satisfy

$$\psi_i(\pm 1) = 0 \quad \text{for } 3 \leqslant i \leqslant p+1.$$

For the case $l=2$ and $p \geqslant 3$, the four nodal shape functions and the remaining $p-3$ internal shape functions given by

$$\begin{aligned} &\psi_1(\tau) = \frac{1}{4}(1-\tau)^2(1+\tau), \quad \psi_2(\tau) = \frac{1}{4}(1-\tau)^2(2+\tau), \\ &\psi_3(\tau) = -\frac{1}{4}(1+\tau)^2(1-\tau), \quad \psi_4(\tau) = \frac{1}{4}(1+\tau)^2(2-\tau), \\ &\psi_i(\tau) = \left(\frac{2i-5}{2}\right)^{1/2} \int_{-1}^{\tau} (\tau-\eta)P_{i-3}(\eta)\mathrm{d}\eta, \quad i = 5,\ldots,p+1. \end{aligned} \tag{2.9}$$

In this case, the internal shape functions satisfy

$$\frac{\mathrm{d}^j\psi_i}{\mathrm{d}\tau^j}(\pm 1) = 0 \quad \text{for } 5 \leqslant i \leqslant p+1 \quad \text{and} \quad j = 0,\ 1. \tag{2.10}$$

The nodal basis functions, ψ_i, $i=1,2,3,4$, in (2.9) also satisfy three of the four conditions in (2.10). For example, using the shape functions in (2.8), any element $u \in L_2[-1,1]$ can be approximated by $u_p \in P_p([-1,1])$, in the form

$$u_p(\tau) = \frac{1-\tau}{2}u(-1) + \frac{1+\tau}{2}u(1) + \sum_{i=3}^{p+1} a_i\psi_i(\tau). \tag{2.11}$$

For approximating the solutions of parabolic problems with the homogeneous Dirichlet boundary condition, the first two terms will be dropped, as $u(-1) = u(1) = 0$. A sequence of triangulations $\{T_h\}_{h>0}$ is called the quasiuniform mesh if

$$\frac{h}{\mathrm{diam}(K)} \leqslant \gamma \quad \text{for all } h > 0, \tag{2.12}$$

with $h = \max_{K\in T_h}\mathrm{diam}(K)$, and for some $\gamma > 0$. $P_p(\Gamma)$ denotes the space of all polynomials of degree $\leqslant p$ defined on Γ. The following is proved by Babuška et al. in [5]. See also [6] by Babuška and Suri on a related discussion. Here Ω_0 denotes a bounded polygonal domain in R^2.

Theorem 2.1. *Let $u \in H^k(\Omega_0)$. Then there exists a sequence $z_p \in P_p(\Omega_0)$, $p = 1, 2, \ldots$, such that, for any $0 \leqslant l \leqslant k$,*

$$\|u - z_p\|_{l,\Omega_0} \leqslant Cp^{-(k-l)}\|u\|_{k,\Omega_0},$$

where C is independent of u and p.

The parameters k and l are not necessarily integral. Their proof relies heavily on the approximation power of the trigonometric polynomials.

With $l = 0$ in Theorem 2.1 and using the usual duality argument, the results in Theorem 2.1 are further improved by Babuška and Suri in [6] (Theorem 2.9), (see also a series of papers by Gui and Babuška [13]), to the *hp*-finite element setting as follows:

Theorem 2.2. *Let T_h be a quasiuniform partition of Ω_0. Then for $k \geqslant 1$, $u \in H^k(\Omega_0)$,*

$$\inf_{v\in S^{p,k}(\omega,T_h)} \|u - v\|_{L_2(\Omega_0)} \leqslant Ch^v p^{-k}\|u\|_{H^k(\Omega_0)},$$

where $v = \min(k, p + 1)$.

The corresponding error estimate in the $\|\cdot\|_{H^k(\Omega_0)}$ is also available in [6].

h-version in the x–y surface variables: First, the *h*-finite element approximation is considered in the x–y variables. Let $z = s(\tau) = \frac{d}{2}\tau$ be the linear transformation of $[-1, 1]$ onto Γ. Now consider the problem of approximating a function $u \in H^k(\omega\times\Gamma)$ by a function from the tensor product space $S_h^r(\omega)\otimes P_p(\Gamma)$, where

$$S_h^r(\omega) = S_h^r(\omega, T_h) = \{u \in L_2(\omega) : u|K \in S^r(K),\ K \in T_h\}. \tag{2.13}$$

For error analysis of *h*-version of the finite element method, the space $S^{p,k}(\omega, T_h)$ defined in (2.7) is not necessary, and the space $S_h^r(\omega)$ of lower dimension can be used to attain the optimal convergence rate. Let $P_h^r : H^2(\omega) \to S_h^r(\omega)$ denote the interpolation projection defined by

$$(P_h^r u)(x, y) = \sum_{i=1}^{r} u\left(x_i^l, y_i^l\right)\varphi_i^l(x, y) \quad \text{for all } (x, y) \in K^l \quad \text{and} \quad u \in H^k(\omega), \tag{2.14}$$

where T_h is a triangulation of ω with $K^l \in T_h$ and $\{(x_i^l, y_i^l)\}_{i=1}^r$ is a set of nodes on K^l with $\varphi_i^l\left(x_j^l, y_j^l\right) = \delta_{ij}$. Also, denote by $Q_p : H^k(\Gamma) \to P_p(\Gamma)$ a projection defined by

$$(Q_p u)(z) = \sum_{i=1}^{p+1} a_i\Psi_i(z) \quad \text{for all } z \in \Gamma, \tag{2.15}$$

where $\Psi_i(z) = \psi_i(s^{-1}(z))$ for each $i \geqslant 1$ where ψ_i are defined, e.g., in (2.8) or (2.9). Recall that for $k = 1$, the constants a_1 and a_2 are known in case of parabolic problems with Dirichlet condition, and it is assumed that a_i, $i \geqslant 3$, in (2.15) are determined by

$$\int_\Gamma \left|u(z) - \sum_{j=3}^{p+1} a_j\Psi_j(z)\right|^2 \mathrm{d}z = \min_{b_j\in R} \int_\Gamma \left|u(z) - \sum_{j=3}^{p+1} b_j\Psi_j(z)\right|^2 \mathrm{d}z. \tag{2.16}$$

From approximation theory [17],

$$\left\|I - P_h^r\right\|_{L_2(\Omega)} = \mathrm{O}(h^r). \tag{2.17}$$

Also $Q_p : L_2(\Gamma) = \to P_p(\Gamma)$, from being the orthogonal projection in the sense described in (2.16) and from Theorem 2.1 that

$$\|I - Q_p\|_{L_2(\Gamma)} = \mathrm{O}(p^{-k}). \tag{2.18}$$

Let

$$\left\|P_h^r \otimes Q_p\right\|_2 \equiv \left\|P_h^r \otimes Q_p\right\|_{L_2(\omega\times\Gamma)} \equiv \sup_{\|u\|_2=1} \left\|(P_h^r \times Q_p)u\right\|_{L_2(\omega\times\Gamma)}.$$

For $u \in L_2(\omega \times \Gamma)$,

$$\left(P_h^r \otimes I\right)u(x,y,z) = \sum_l \sum_{i=1}^{r} u\left(x_i^l, y_i^l, z\right)\varphi_i^l(x,y),$$

and

$$(P_h^r \otimes Q_p)u(x,y,z) = \sum_{j=1}^{p+1}\left\{\sum_l \sum_{i-1}^{r} \varphi_i^l(x,y)\right\}a_j\Psi_j(z),$$

where a_j depends upon u and obtained according to (2.16). First, approximation order under L_2 operator norm of $P_h^r \otimes Q_p$ for $P_h^r \otimes I$ is established.

Lemma 2.3. *For $P_h^r : H^k(\omega) \to S_h^r(\omega), 0 \leqslant r \leqslant k$, and $Q_p : H^k(\Gamma) \to P_p(\Gamma)$ defined respectively in (2.14) and (2.15),*

$$\left\|P_h^r \otimes I - P_h^r \otimes Q_p\right\|_2 \leqslant Cp^{-k},$$

where C is independent of p.

Proof

$$\begin{aligned}
\left\|P_h^r \otimes I - P_h^r \otimes Q_p\right\|_2 &\equiv \sup_{\|u\|_2=1} \left\|\left(P_h^r \otimes I - P_h^r Q_p\right)u\right\|_2 \\
&= \sup_{\|u\|_2=1}\left\{\sum_l \int_{K^l}\int_{\Gamma}\left|\sum_{i=1}^{r} u\left(x_i^l, y_i^l, z\right)\varphi_i^l(x,y) - \sum_{j=1}^{p+1}\sum_{i=1}^{r}\varphi_i^l(x,y)a_j\Psi_j(z)\right|\mathrm{d}z\,\mathrm{d}x\,\mathrm{d}y\right\}^{1/2} \\
&\leqslant \sup_{\|u\|_2=1}\left\{\sum_l \int_{K^l}\int_{\Gamma}\left|\sum_{i=1}^{r}\varphi_i^l(x,y)\left(u\left(x_i^l, y_i^l, z\right) - \sum_{j=1}^{p+1}a_j\Psi_j(z)\right)\right|^2\mathrm{d}z\,\mathrm{d}x\,\mathrm{d}y\right\}^{1/2} \\
&\leqslant \sup_{\|u\|_2=1}\sum_{i=1}^{r}\left\{\sum_l \int_{K^l}\int_{\Gamma}\left|\varphi_i^l(x,y)\right|^2\left|u\left(x_i^l, y_i^l, z\right) - \sum_{j=1}^{p+1}a_j\Psi_j(z)\right|^2\mathrm{d}z\,\mathrm{d}x\,\mathrm{d}y\right\}^{1/2} \\
&\quad \text{by Minkowski inequality} \\
&\leqslant \sup_{\|u\|_2=1} r\max_{1\leqslant i\leqslant r}\left\{\sum_l \int_{K^l}\left|\varphi_i^l(x,y)\right|^2\left|u\left(x_i^l, y_i^l, z\right) - \sum_{j=1}^{p+1}a_j\Psi_j(z)\right|^2\mathrm{d}z\,\mathrm{d}x\,\mathrm{d}y\right\}^{1/2} \\
&\leqslant C\sup_{\|u\|_{2=1}}\max_{1\leqslant i\leqslant r}\left\{\int_{\Gamma}\left|u\left(x_i^l, y_i^l, z\right) - \sum_{j=1}^{p+1}a_j\Psi_j(z)\right|^2\mathrm{d}z\right\}^{1/2} \\
&\quad \text{where } M \equiv \sum_l \int_{K^l}\left|\varphi_i^l(x,y)\right|^2\mathrm{d}x\,\mathrm{d}y \\
&\leqslant Cp^{-k}, \quad \text{by Theorem 2.1.} \qquad \square
\end{aligned}$$

Similarly, the following lemma will be useful.

Lemma 2.4. *Let $P_h^r : H^k(\omega) \to S_h^r$, with $0 \leqslant r \leqslant k$ and $Q_p : H^k(\Gamma) \to P_p(\Gamma)$. Then*

$$\left\|I \otimes Q_p - P_h^r \otimes Q_p\right\|_2 \leqslant Ch^r,$$

where C is independent of r.

Proof

$$\begin{aligned}
\left\|I\otimes Q_p - P_h^r\otimes Q_p\right\|_2 &\equiv \sup_{\|u\|2=1}\left\|\left(I\otimes Q_p - P_h^r\otimes Q_p\right)u\right\|_2 \\
&= \sup_{\|u\|2=1}\left\{\sum_{K^l\in T_h}\int_{K^l}\int_{\Gamma}\left|\sum_{j=1}^{p+1}\alpha_j(x,y)\Psi_j(z) - \sum_{j=1}^{p+1}\left(\sum_{i=1}^{r}\alpha_j\left(x_i^l,y_i^l\right)\varphi_i(x,y)\right)\Psi_j(z)\right|^2 \mathrm{d}z\mathrm{d}x\mathrm{d}y\right\}^{1/2} \\
&\quad \text{where } \sum_{j=1}^{p+1}\alpha_i(x,y)\Psi_j(z) \text{ is the best } L_2(\Gamma) \text{ approximation of } u(x,y,\cdot) \\
&\leqslant \sup_{\|u\|2=1}\left\{\sum_{K^l\in T_h}\int_{K^l}\int_{\Gamma}\left|\sum_{j=1}^{p+1}\left\{\alpha_j(x,y) - \sum_{i=1}^{r}\alpha_j\left(x_i^l,y_i^l\right)\varphi_i(x,y)\right\}\Psi_j(z)\right|^2 \mathrm{d}z\mathrm{d}x\mathrm{d}y\right\}^{1/2} \\
&\leqslant \sup_{\|u\|2=1}\sum_{j=1}^{p+1}\left\{\sum_{K^l\in T_h}\int_{K^l}\int_{\Gamma}\left|\alpha_j(x,y) - \sum_{i=1}^{r}\alpha_j\left(x_i^l,y_i^l\right)\varphi_i(x,y)\right|^2\left|\Psi_j(z)\right|^2 \mathrm{d}z\mathrm{d}x\mathrm{d}y\right\}^{1/2} \\
&\leqslant C\left\{\sum_{K^l\in T_h}\int_{K^l}\left|\alpha_j(x,y) - \sum_{i=1}^{r}\alpha_j\left(x_i^l,y_i^l\right)\varphi_i(x,y)\right|^2\left|\right| \mathrm{d}x\mathrm{d}y\right\}^{1/2} \\
&\leqslant Ch^r, \quad \text{provided that } \alpha_j\in H^r(\omega),
\end{aligned}$$

where the last inequality follows from a well-known result of the approximation power of piecewise polynomials [17]. □

Using Theorems 2.1 and 2.2, we obtain the following theorem which provides an error estimate for approximating an element in $H^k(\omega\times\Gamma)$ by elements from $S_h^r(\omega)\otimes P_p(\Gamma)$. The result will be used in the next section when the formulation of error estimate of the modified *h–p* discontinuous Galerkin finite element method for approximating the solution of the parabolic problem (1.1) is established.

Theorem 2.5. *Let $u\in H^k(\omega\times\Gamma)$. Then there exists $u^*\in S_h^r(\omega)\otimes P_p(\Gamma)$ such that for $0\leqslant r\leqslant k, p>0$,*

$$\|u-u^*\|_{L_2(\omega\times\Gamma)} = \mathrm{O}(h^r + p^{-k}).$$

Proof. Define $u^* = (P_h^r\otimes Q_p)u$. Then, using Lemmas 2.3 and 2.4

$$\begin{aligned}
\|u-u^*\|_2 &= \|u-(P_h^r\otimes Q_p)u\|_2 = \|u-(I\otimes Q_p)u + (I\otimes Q_p)u - (P_h^r\otimes Q_p)u\|_2 \\
&\leqslant \|u-(I\otimes Q_p)u\|_2 + \|(I\otimes Q_p)u - (P_h^r\otimes Q_p)u\|_2 = \mathrm{O}(h^r + p^{-k}). \qquad \square
\end{aligned}$$

hp-version in the x–y surface variables: Now we incorporate the *hp*-version of approximation technique in the *x–y* coordinates. The goal is to approximate a function $u\in H^k(\omega\times\Gamma)$ from the tensor product space $S^{p_1,k}(\omega,T_h)\otimes P_{p2}(\Gamma)$ for nonnegative integers p_1 and p_2. Analysis is similar to the one given in Lemmas 2.3 and 2.4 and therefore is not given. Using Theorem 2.2, it can be seen easily that

Theorem 2.6. *Let $u\in H^k(\omega\times\Gamma)$. Then there exists $u^*\in S^{p_1,k}(\omega,T_h)\otimes P_p^2(\Gamma)$,*

$$\|u-u^*\|_{L_2(\omega\times\Gamma)} = \mathrm{O}\left(h^v p_1^{-k} + p_2^{-k}\right).$$

where $v = \min(k, p_1+1)$ and $h = \max_{K\in T_h}\mathrm{diam}(K)$, with T_h a triangulation of ω.

Remark. Let $N(p)\equiv\frac{(p+1)(p+2)}{2}$. Note that numbers of the degrees of freedom of $S^{p_1,k}(\omega,T_h)$ and $P_{p2}(\Gamma)$ are $M(h)N(p_1)$ and $N(p_2)$ respectively. Since a single element through the thickness is used because of the specific structural consideration in this paper, we cannot expect the total error to decrease by letting the diameter

$h \to 0$, i.e., by letting the size of surface elements decrease to 0. The second error term would quickly dominates the overall performance of approximation in that case. In order for both of the error terms in Theorem 2.6 to decrease consistently, note that $N(p) = O(p^2)$ and $h = O(M(h)^{-1})$. Thus the number of surface elements $M(h)$ and the corresponding degree p_1 of polynomials should be selected so as to maintain

$$M(h)^{-k} N(p_1)^{-\frac{k}{2}} \simeq N(p_2)^{-\frac{k}{2}}. \tag{2.19}$$

Eq. (2.19) not only describes the consistent error estimates between the two terms but also indicates the consistent workloads between the surface and the through the thickness approximations.

3. Discontinuous Galerkin method

In this section, the discontinuous Galerkin (DG) method for problem (1.1) is developed. The discontinuity is introduced in time, which allows computation to march forward in time. This, when compared with the standard continuous Galerkin method, presents an enormous saving in size of computation. The DG finite element method for parabolic partial differential equations was studied in a series of papers by Erikson, Johnson and Larsson, [8–12]. In these papers, the convergence in time of h-finite element DG method is established for solutions which are smooth. More specifically, when the solutions are approximated by polynomials of degree r, then the algebraic error estimate of $O(\Delta t^{r+1})$ as $\Delta t \to 0$ is obtained. However, in many parabolic partial equations, solutions exhibit singularities at $t = 0$ due to the initial conditions. In a recent paper [1], the present authors established a graded time discretization scheme that captures the transient solution to optimal precision. The graded time mesh is selected by assuming that $\|u_t\|_2$ is weakly singular. A similar study of the graded time meshes is reported recently by Schötzau and Schwab [16]. They derive a set of graded time partition points by considering an incompatibility between initial and boundary conditions. It is demonstrated in [1] that the time discretization based upon $\|u_t\|_2$ provides more relaxed distribution of partition points. The paper of Schötzau and Schwab goes on to describe the p-finite element in time and obtain an exponential convergence in spite of a singular transient phase of the solution. We will not discuss the p-finite element in time in this paper. It will be taken up in [2] in which the complete p-finite element for parabolic problems is discussed.

We begin by recalling several results from [2] that are pertinent to the present paper. The following conditions will be assumed. Recall from Section 2 that $\Omega = \omega \times \Gamma$. Let (h, T, S) denote a finite element discretization satisfying

1. h is a positive function in $C^1(\overline{\Omega})$ such that
   $$|\nabla h(x)| \leqslant M \quad \text{for all } x \in \overline{\Omega} \text{ and for some } M > 0.$$
2. $T = \{K\}$ is a set of triangular subdomain of ω with each triangular element having diameter h_K such that
   $$c_1 h_K^2 \leqslant \int_K \mathrm{d}x \quad \text{for all } K \in T,$$
   and associated with the function h through
   $$c - 1 h_K \leqslant h(x) \leqslant c_2 h_K \quad \text{for all } x \in K,\ K \in T,$$
   where $c_1 > 0$, $c_2 > 0$.
3. S is the set of all continuous functions on $\overline{\Omega}$ which are polynomials of order r in $x = (x_1, x_2)$ on each $K \in T$ and vanish on $\partial\omega$ as well as which are polynomials of order p in the z-variable in Γ.

For the discontinuous Galerkin method for (1.1), we partition R^+ as $0 = t_0 < t_1 < \cdots < t_n < \cdots$ where we let $I_n \equiv (t_{n-1}, t_n]$ with $k_n \equiv t_n - t_{n-1}$. For each time interval, with q a nonnegative integer, We let

$$W_{hp}^q \equiv \{v : R^+ \to V_{hp} : v|_{I_n} \in P_q(I_n),\ n = 1, \ldots, N\},$$

where

$$V_{hp} = \left\{ \begin{array}{l} \text{the space of all functions } u^* \in S_h^r(\omega) \otimes P_p(\Gamma) \\ \text{or } u^* \in S_p^k(\omega, T_h) \otimes P_p(\Gamma) \text{ such that} \\ h = \max_{K \in T_h} \operatorname{diam}(K) \quad \text{where } T_h \text{ is a triangulation of } \omega \end{array} \right\}$$

and

$$P_q(I_n) = \left\{ v(t) = \sum_{i=0}^{q} v_i t^i : v_i \in V_{hp} \right\}.$$

The discontinuous Galerkin method is defined as follows:

Find U such that for $n = 1, 2, \ldots$, with $\Omega = \omega \otimes \Gamma$, $U|_{\Omega \times I_n} \in W_{hp}^q$ and

$$\int_{I_n} \{(U_t, v) + a(U, v)\} \mathrm{d}t + \left([U]_{n-1}, v_{n-1}^+\right) = \int_{I_n} (f, v) \mathrm{d}t \quad \text{for all } v \in W_{hp}^q, \tag{3.1}$$

where $[w]_n = w_n^+ - w_n^-$, $w_n^{+(-)} = \lim_{s \to 0^{+(-)}} w(t_n + s)$, $U_0^- = u_0$, $(u, v) = \int_\Omega u(x)v(x)\mathrm{d}x$ and $a(u, v) = (\nabla U, \nabla v)$. The smoothness of $\|u_t\|_{L_2(\Omega)}$ is subject to the initial condition as well as to the boundary condition. For example, if we take in (1.1), $u_0(x) = \pi - x$, $f(x, t) \equiv 0$ and $\Omega = (0, \pi)$, then the actual solution of the corresponding problem is given by

$$u(x, t) = \sum_{j=1}^{\infty} u_j^0 \mathrm{e}^{-j^2 t} \sin(jx),$$

where

$$u_j^0 = \frac{2}{\pi} \int_0^\pi (\pi - x) \sin(jx) \mathrm{d}x = \frac{2}{\pi} \left\{ \frac{\pi}{j} - \frac{1}{j^2} \sin^2 j\pi \right\} = \mathrm{O}\left(\frac{1}{j}\right).$$

In the following, C's denote generic constants whose values change as they appear. Now,

$$\|u_t(t)\|_2^2 = \|u_t(t)\|_{L_2(\Omega)}^2 = \sum_{j-1}^{\infty} C j^2 \mathrm{e}^{-2j^2 t} = \frac{\mathrm{d}}{\mathrm{d}t} \sum_{j=1}^{\infty} C \mathrm{e}^{-2j^2 t}. \tag{3.2}$$

The last equality in (3.2) is justified because of the uniform convergence of $\sum_{j=1}^{\infty} C j^2 \mathrm{e}^{-2j^2 t}$ with respect to t. Now using the fact that $\int_0^\infty \mathrm{e}^{-x^2} \mathrm{d}x < \infty$, a simple change of variables (say, $y = j\sqrt{2t}$) will show that the last expression in (3.2) is $\frac{\mathrm{d}}{\mathrm{d}t} C t^{-1/2}$, which leads to

$$\|u_t(t)\|_2 = \mathrm{O}(t^{-3/4}).$$

A similar argument shows that if $u_j^0 = O(\frac{1}{j^2})$ for some initial value function $u_0(x)$, then $\|u_t(t)\|2 = \mathrm{O}(t^{-1/4})$. This case arises when $u_0(x) = \min(x, \pi - x)$ for $x \in (0, \pi)$. If u_j^0 decays faster than $j^{-2.5}$ as $j \to \infty$, then $\|u_t(t)\|_2$ will be bounded as $t \to 0$. An initial phase for small t is the well-known initial transient for parabolic problems. It is the case that the smoothness of the solutions of parabolic problems vary significantly in space and time with initial transients where highly oscillatory components of the solution are decaying rapidly. Therefore, in order for numerical methods for parabolic problems to be successful, it is imperative that the methods take a careful account of time and space discretization scheme so as to capture the transient solutions. An adaptive time step control scheme was established by Eriksson and Johnson in [9]. Time steps k_n are defined by controlling the size of

$$\min_{j \leqslant q+1} k_n^j \left\| u_t^{(j)} \right\|_{I_n},$$

where q is the order of spline used in time and $u_t^{(1)} = u_t$, $u_t^{(2)} = u_{tt}$, $u_t^{(3)} = \Delta u_{tt}$ and $\|w\|_{I_n} = \max_{t \in I_n} \|w(t)\|_2$. Note that the method of Eriksson and Johnson requires some estimates concerning $\|u_{tt}\|_{I_n}$ and $u_t^{(3)} = \Delta u_{tt}$ to achieve the second and the third-order convergence in time. The approach given in [1] provides convergence of any order in time for the discontinuous Galerkin method by examining only the behavior of $\|u_t\|_2$.

For $0 < \alpha < 1$ and q a nonnegative integer, define $Q \equiv \frac{q+1}{1-\alpha}$. For a positive integer N and $T > 0$, define

$$t_n^* = \left(\frac{n}{N}\right)^Q \quad n = 0, 1, \ldots, N$$

and

$$t_n = t_n^* T. \tag{3.3}$$

We let $I_n = (t_{n-1}, t_n]$, $n = 1, 2, \ldots, N$. Let k_n denote the length of I_n so that

$$k_n = \left[\left(\frac{n}{N}\right)^Q - \left(\frac{n-1}{N}\right)^Q\right] T, \quad n = 1, 2, \ldots, N.$$

Note that

$$k_n \leqslant Q\left[\frac{n}{N}\right]^{Q-1} \frac{1}{N} T \quad \text{by the mean value theorem,}$$

hence

$$k_n \leqslant C \frac{1}{N^{q+1}}, \tag{3.4}$$

where C is a constant independent of n. The solution $u(x, t)$ of (1.1) is then approximated in t over each I_n by a polynomial of degree q. For example, with $q = 1$, let $I_n^i w$ denote the linear interpolatory projection of $w \in H_0^2$ in time onto W_{hk}, viz,

$$I_n^i w(x, t) = \frac{t_n - t}{k_n} w(x, t_{n-1}) + \frac{t - t_{n-1}}{k_n} w(x, t_n) \quad \text{for each } t \in I_n.$$

Note that I_n^i, considered as an operator defined on H_0^2 is bounded with respect to the norm $\|\cdot\|_{\infty, I_n}$ where

$$\|w(t)\|_{\infty, I_n} \equiv \max_{t \in I_n} \|w(t)\|_{L_\infty(\Omega)}.$$

Since Ω is assumed to be of bounded domain, I_n^i is bounded with respect to $\|\cdot\|_{I_n}$ also. As was the case with the L_2 projection, I_n^i equals the identity on polynomials of degree $\leqslant 1$. Expanding $u(x, t)$ in Taylor series with respect to t at t_n to the first or to the second order, we obtain, respectively, for each $n = 1, 2, \ldots, N$,

$$\left\|u - I_n^i u\right\|_{I_n} \leqslant \int_{I_n} \|u_t(s)\|_2 \mathrm{d}s. \tag{3.5}$$

Lemma 3.1. *Let $0 < \alpha < 1$, q a nonnegative integer and $T > 0$, we assume that t_n, $n = 1, \ldots, N$ are defined by (3.3). Then*

$$\max_{n \leqslant N} \int_{I_n} s^{-\alpha} \mathrm{d}s \leqslant C_n \frac{1}{N^{q+1}},$$

where C_n is a constant independent on N.

Lemma 3.2. *Let t_n and k_n be defined by (3.3). Then*

$$\left(1 + \log \frac{t_n}{k_n}\right)^{1/2} \leqslant \sqrt{2} \quad \textit{for each } n = 0, 1, \ldots, N.$$

Lemma 3.2 is used to guarantee the stability of the discontinuous Galerkin method. In the remainder of this paper, we illustrate the current 'modified' *hp*-finite element method by assuming the *h*-version in the surface x–y variables using the linear splines. Also we illustrate the cases for constant as well as linear degree in time approximation. Let $\{(x_i, y_i)\}_{i=1}^M$ is the set of nodal points which are the interior vertices of K in T_h. Let φ_j

be the linear spline basis element defined by $\varphi_j(x_i, y_i) = \delta_{ij}$, for $i,j = 1,\ldots,M$. The superscript l used in (2.2) will be dropped. For application of higher-order spline basis, more nodal points are required over each K. The solution u of (1.1) is approximated by ($t > 0$)

$$u(x,y,z,t) \sim \sum_{j=1}^{M} \left\{ u\left(x_j, y_j, -\frac{\Delta}{2}, t\right)\left(\frac{1}{2} - \frac{z}{\Delta}\right) + u\left(x_j, y_j, \frac{\Delta}{2}, t\right)\left(\frac{z}{\Delta} - \frac{1}{2}\right) + \sum_{i=3}^{p+1} a_i^j(t)\psi_i(z)\varphi_j(x,y). \right. \tag{3.6}$$

Note that $u\left(x_j, y_j, \mp\frac{\Delta}{2}, 0\right)$, for $j = 1,\ldots,M$ are known from the initial condition. Also, for $t > 0$, the boundary values $u\left(x_j, y_j, \mp\frac{\Delta}{2}, t\right)$ are given. As $u(\bar{x}, t) = 0$, for $\bar{x} \in \partial\Omega$, $t \in R^+$ in (1.1) and (3.6) simplifies to

$$u(x,y,z,t) \sim \sum_{j=1}^{M} \sum_{i=3}^{p+1} a_i^j(t)\psi_i(z)\varphi_j(x,y). \tag{3.7}$$

At each time level t_n, we approximate $u(x,y,z,t_n) = u(\bar{x}, t)$ by

$$U^n = U^n(\bar{x}) = U(\bar{x}, t_n) = \sum_{j=1}^{M} \sum_{i=3}^{p+1} a_i^j(t_n)\psi_i(z)\varphi_j(x,y), \quad n = 0, 1, \ldots, N. \tag{3.8}$$

To start the DG finite element method, we first require $a_i^j(t_0)$ and they are determined from $u_0(\bar{x})$. More specifically, for each $j = 1,\ldots,M$, since $u_0(x_j, y_j, z, t_0) \sim U^0(x_j, y_j, z, t_0) = \sum_{i=3}^{p+1} a_i^j(t_0)\psi_i(z)$, $M(p-1)$ many $a_i^j(t_0)$ are found by solving

$$\sum_{k=3}^{p+1} \int_{-\frac{\Delta}{2}}^{\frac{\Delta}{2}} \psi_k(z)\psi_i(z)\mathrm{d}z \cdot a_k^j(t_0) = \int_{-\frac{\Delta}{2}}^{\frac{\Delta}{2}} u(x_j, y_j, z, 0)\psi_i(z)\mathrm{d}z \quad \text{for } i = 3,\ldots,p+1.$$

Now, Eq. (3.1) can be formulated as follows:

For $n = 1,2,\ldots,N$, given $U^{n-1,-}$, find $U \equiv U|_{I_n} = \in P_q(I_n)$ such that

$$\int_{I_n} [(U_t, v) + a(U, v)]\mathrm{d}t + (U^{n-1,+}, v^{n-1,+}) = \int_{I_n} (f, v)\mathrm{d}t + (U^{n-1,-}, v^{n-1,+}), \tag{3.9}$$

for all $v \in P_q(I_n)$ where $U^{0,-} = u_0$.

For a special case, consider $q = 0$, i.e., constant in time. As $U^n = U^{n,-} = U^{n-1,+}$ in this case, (3.9) reduces to

$$(U^n - U^{n-1}, v) + k_n a(U^n, v) = \int_{I_n} (f, v)\mathrm{d}t, \tag{3.10}$$

for all $v \in P_0(I_n)$ and $n = 1,2,\ldots,N$. With (3.7) and (3.10) becomes for each $n = 1,2,\ldots$

$$\sum_{j=1}^{M} \sum_{i=3}^{p+1} a_i^j(t_n)[(\psi_i\varphi_j, \psi_\alpha\varphi_\beta) - k_n a(\psi_i\varphi_j, \psi_\alpha\varphi_\beta)] = \sum_{j=1}^{M} \sum_{i=3}^{p+1} a_i^j(t_{n-1})(\psi_i\varphi_j, \psi_\alpha\varphi_\beta) + (f, \psi_\alpha\varphi_\beta),$$

for each $\alpha = 3,\ldots,p+1$; $\beta = 1,\ldots,M$.

For $q = 1$, we let $U|_{I_n} = \Phi_n(\bar{x}) + \frac{t-t_{n-1}}{k_n}\Psi_n(\bar{x})$ where $\Phi_n = \sum_{j=1}^{M}\sum_{i=3}^{p+1} a_i^{\Phi,j}(t_n)\psi_i(z)\varphi_j(x,y)$ and $\Psi_n = \sum_{j=1}^{M}\sum_{i=3}^{p+1} a_i^{\Psi,j}(t_n)\psi_i(z)\varphi_j(x,y)$. As $U^{n-1+} = \Phi_n$ and $U^{n-1,+} = \Phi_{n-1} + \Psi_{n-1}$ (3.9) becomes

$$\int_{I_n} \left[\frac{1}{k_n}(\bar{\Psi}_n, v) + a\left(\Phi_n + \frac{t - t_{n-1}}{k_n}\Psi_n, v\right)\right]\mathrm{d}t + \left(\Phi_n, v_+^{n-1}\right) = \int_{I_n} (f, v)\mathrm{d}t + (\Phi_{n-1} + \Psi_{n-1}, v_+^{n-1}), \tag{3.11}$$

for all $v \in P_1(I_n)$. By taking $v = \psi_\alpha\varphi_\beta$ and $\frac{t-t_n}{k_n}\psi_\alpha\varphi_\beta$, (3.11) reduces to the following linear equations for $2M(p-1)$ unknowns $a_i^{\Phi,j}(t_n)$ and $a_i^{\Psi,j}(t_n)$:

$$\sum_{j=1}^{M} \sum_{i=3}^{p+1} a_i^{\Phi,j}(t_n)\{(\varphi_j\psi_i, \varphi_\beta\psi_\alpha) + k_n a(\varphi_j\psi_i, \varphi_\beta\psi_\alpha)\} + \sum_{j=1}^{M} \sum_{i=3}^{p+1} a_i^{\Psi,j}(t_n)\{(\varphi_j\psi_i, \varphi_\beta\psi_\alpha) + k_n a(\varphi_j\psi_i, \varphi_\beta\psi_\alpha)\}$$

$$= \int_{I_n} (f(t), \varphi_\beta\psi_\alpha) + \sum_{j=1}^{M} \sum_{i=3}^{p+1} \left[a_i^{\Phi,j}(t_{n-1}) + a_i^{\Phi,j}(t_{n-1})\right](\varphi_j\psi_i, \varphi_\beta\psi_\alpha), \quad \alpha = 3,\ldots,p+1;\ \beta = 1,\ldots,M,$$

$$\sum_{j=1}^{M}\sum_{i=3}^{p+1} a_i^{\Phi,j}(t_n)\frac{k_n}{2}a(\varphi_j\psi_i,\varphi_\beta\psi_\alpha)+k_n a(\varphi_j\psi_i,\varphi_\beta\psi_\alpha)$$

$$+\sum_{j=1}^{M}\sum_{i=3}^{p+1} a_i^{\Psi,j}(t_n)\left\{\frac{1}{2}(\varphi_j\psi_i,\varphi_\beta\psi_\alpha)+\frac{k_n}{3}a(\varphi_j\psi_i,\varphi_\beta\psi_\alpha)\right\}$$

$$=\frac{1}{k_n}\int_{I_n}(t-t_{n-1})(f(t),\varphi_\beta\psi_\alpha)\mathrm{d}t,\quad \alpha=3,\ldots,p+1;\ \beta=1,\ldots,M.$$

The following theorem can be proved by minor modifications to the proof of Theorem 1.1, [9] and by making use of Theorem 2.5. The present theorem is described for $\Omega=\omega\otimes\Gamma\subseteq R^3$.

Theorem 3.3. *Suppose that there is a constant γ such that the time steps k_n satisfy $k_n\leqslant\gamma k_{n+1}$, $n=1,\ldots,N-1$ and let U_n denote the solution of (3.9) approximating u at t_n. Here u is approximated by a polynomial of degree $q\geqslant 0$ over each I_n for $n=1,\ldots,N-1$, and $u(\cdot,\cdot,\cdot,t)$ is approximated by an element from $S_h^r(\omega)\otimes P_p(\Gamma)$ for each $t\in R^+$, where ω is a polygonal domain in R^2. Then there is a constant C depending only on γ and a constant β, where $\rho_K\geqslant\beta h_K$ and ρ_K is the diameter of the circle inscribed in K for all $K\in T_h$, such that for $n=1,2,\ldots,N$,*

$$\|u(t_n)-U_n\|_2\leqslant C\left(1+\log\frac{t_n}{k_n}\right)^{1/2}\left\{\max_{m\leqslant n}\left\|u-I_n^i u\right\|_{I_m}+h^2\left\|D_{xy}^2u\right\|_{I_n}+p^{-k}\|u\|_{I_n,H^k(\Gamma)}\right\},\tag{3.12}$$

where $\|w\|_{I_n,H^k(\Gamma)}=\max_{t\in I_n}\|w(t)\|_{H^k(\Gamma)}$ and D_{xy}^2 denotes the second-order derivative with respect to x and y variables.

Lemma 3.2 guarantees that the current DG finite element method with the graded temporal meshes defined in (3.3) is a stable scheme. Also Lemma 3.1 provides a bound for the term $\max_{n\leqslant m}\left\|u-I_n^i u\right\|I_m$ in (3.12) provided that $\|u_t\|_2=\mathrm{O}(t^{-\alpha})$ as $t\to 0$. Theorem 2.5 is used to control the last two error terms in Theorem 3.3. In summary, we obtain the following theorem which utilizes the traditional h-finite element in the surface variables.

Theorem 3.4. *Suppose that $u\in H^k(\omega\times\Gamma)$ and $\|u_t\|_2=\mathrm{O}(t^{-\alpha})$ for $0<\alpha<1$ and that the time partition points t_n are taken according to (3.3). Let U_n denote the solution of (3.9) approximating u at t_n. Also assume that u is approximated in time variable by a polynomial of degree $q\geqslant 0$ over each I_n for $n=1,\ldots,N-1$, and $u(\cdot,\cdot,\cdot,t)$ is approximated by an element from $S_h^2(\omega)\otimes P_p(\Gamma)$ for each $t\in R^+$, where ω is a polygonal domain in R^2. Then*

$$\|u(t_n)-U_n\|_2=\mathrm{O}(N^{-(q+1)}+h^2+p^{-k}).$$

If higher-order $r>2$ splines are used in $S_h^r(\omega)$, then the second term in the error can be replaced by h^r provided $\left\|D_{xy}^r u\right\|_{I_n}$ is bounded.

In the case of the *hp*-finite element approximation for the surface variables, Theorem 2.6 is now used to establish the following.

Theorem 3.5. *Suppose that $u\in H^k(\omega\times\Gamma)$ and $\|u_t\|_2=\mathrm{O}(t^{-\alpha})$ for $0<\alpha<1$ and that the time partition points it t_n are taken according to (3.3). Let U_n denote the solution of (3.9) approximating u at t_n. Also assume that u is approximated in time variable by a polynomial of degree $q\geqslant 0$ over each I_n for $n=1,\ldots,N-1$, and $u(\cdot,\cdot,\cdot,t)$ is approximated by an element from $S^{p_1,k}(\omega,T_h)\otimes P_{p2}(\Gamma)$ for each $t\in R^+$, where ω is a polygonal domain in R^2. Then*

$$\|u(t_n)-U_n\|_2=\mathrm{O}(N^{-(q+1)}+h^v p_1^{-k}+p_2^{-k}),$$

where $v=\min(k,p_1+1)$.

Numerical experiments of the presently proposed 'modified' *h–p* finite element method for parabolic equations will be reported elsewhere in future.

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