ARTICLE IN PRESS

Available online at www.sciencedirect.com

ScienceDirect

Applied Mathematics and Computation xxx (2008) xxx–xxx

APPLIED MATHEMATICS AND COMPUTATION

www.elsevier.com/locate/amc

Numerical experiments using hierarchical finite element method for nonlinear heat conduction in plates

Hideaki Kaneko [a,*,1], Kim S. Bey [b], Yongwimon Lenbury [c], Puntip Toghaw [c]

[a] Department of Mathematics and Statistics, Old Dominion University, Norfolk, VA 23529-0077, United States
[b] Thermal Structure Branch, Structure Division, NASA Langley Research Center, Hampton, VA 23681, United States
[c] Department of Mathematics, Mahidol University, Bangkok, Thailand

Abstract

In this paper, we consider a nonlinear hierarchical finite element method for heat conduction problems over two- or three-dimensional plates. Problems considered are nonlinear because the heat conductivity parameter depends upon the temperature itself. This paper explores a new technique recently proposed by the first author which transforms a nonlinear parabolic problem to a linear problem at the discrete level. We present several numerical examples which demonstrate the efficiency of the current technique.

Keywords: Discontinuous Galerkin method; Nonlinear parabolic equations

1. Introduction

In this paper, we report on some recent numerical experiments to approximate the solution of a class of nonlinear heat conduction problems proposed over two- or three-dimensional plates. The problem is nonlinear because the heat conductivity parameter depends upon the temperature itself. This, of course, describes the natural state of heat conduction for most materials as their capacities to conduct heat varies with temperature. To deal with nonlinearity, a new technique which was originally proposed in [3] is used in this paper to transform the problem to a linear problem at the discrete level. The technique is tested on several examples in the last section to demonstrate the efficiency of the technique. The general approach which was taken in this paper is the discontinuous Galerkin finite element method (DGFEM). Discontinuity is applied to the time variable. Unlike the continuous Galerkin finite element method where the entire time domain must be treated simultaneously, DGFEM allows computation to march forward in time. This reduces the size of computation. As to the spatial discretization, a hierarchical modelling is done through the thickness of plate. The use of appropriate basis functions associated with a hierarchical modelling transforms the heat conduction problem to a

* Corresponding author.
E-mail address: hkaneko@odu.edu (H. Kaneko).
[1] This author is supported by NASA-Grant NNL04AA58G.

0096-3003/$ - see front matter © 2008 Published by Elsevier Inc.
doi:10.1016/j.amc.2007.12.062

hierarchy of dimensionally reduced plate problems at each time step. Hierarchical modelling technique is conceptually simple devoid of three-dimensional elements and it has been used extensively for various plates problems in engineering. Most of the research on the hierarchical modelling are done for steady-state heat problems. In 1981, a rigorous mathematical framework of the hierarchical medelling method was established by Babuska and Vogelius in [15,16]. Subsequently, their results were extended by Schwab to those problems with boundary layers, [11,12]. It was established in these papers that by increasing the order of approximation in an appropriate region near the boundary, the optimal order of convergence by the hierarchical modelling is restored in orthotropic approximation as the thickness $d \to 0$. We test our method on four numerical examples. First, we consider a two-dimensional plate problem in which compatible data is assumed for boundary conditions so that the solution is uniformly smooth throughout the region at any time $t > 0$, free of boundary layers. Second, we test our method on a similar problem in the three-dimensional setting. Third, we test the current iterative method on a two-dimensional heat conduction problem with incompatible initial and boundary data. Finally, as the fourth example, we consider the hierarchical modelling technique for a plate with two layers. This requires a construction of a special class of basis functions.

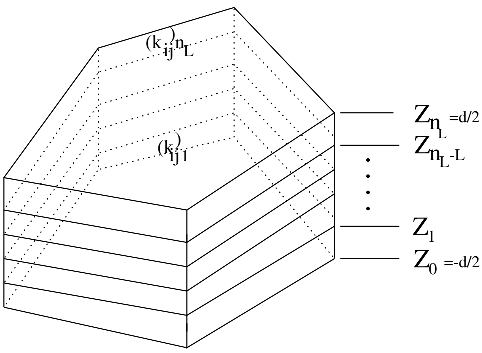
The dimension reduction technique of the hierarchical modelling for a steady-state nonlinear heat conduction was investigated by Jensen and Babuska [8] and by Jensen [9]. But, its application to nonlinear parabolic equations which is done in this paper appears to be new. For discretization over surface of a plate, the standard h-finite element approximation is employed. In Section 2, the discontinuous Galerkin method based on the hierarchical modelling is described. New approach which was taken in this paper to deal with the nonlinear conduction term is also described in this section. Some results from the papers [4–6] by Eriksson and Johnson and [7] by Eriksson et al. are used to describe the convergence of the current method. The aforementioned numerical examples are provided in Section 3.

2. DGFEM

The following model problem of nonlinear parabolic type is considered:
Find u such that

$$\begin{aligned} & u_t - \operatorname{div}(a(u)\nabla u) = f \quad \text{in } \Omega,\ \ t \in R^+, \\ & u(x,t) = 0, \quad x \in \Gamma,\ \ t \in R^+, \\ & \frac{\partial u}{\partial n}(x,t) = f^{\pm}, \quad x \in R_{\pm},\ \ t \in R^+, \\ & u(x,0) = u_0(x), \quad x \in \Omega, \end{aligned} \tag{2.1}$$

where $\Omega = \omega \times \left(-\frac{d}{2}, \frac{d}{2}\right)$ is a closed and bounded set in R^3 with $\Gamma = \partial\omega \times \left(-\frac{d}{2}, \frac{d}{2}\right)$, $R_{\pm} = \omega \times \{\pm\frac{d}{2}\}$ and $R^+ = (0, \infty)$. The functions f and u_0 are given data. Problem (2.1) describes a heat conduction problem with heat conductivity function a affecting the temperature u. An interesting special case of the region Ω is the multilayered plate as shown below: Here the thickness domain $\left[-\frac{d}{2}, \frac{d}{2}\right]$ is layered and layers are partitioned by $Z_0, Z_1, \ldots, Z_{n_L}$. Also each layer $[Z_\ell, Z_{\ell+1}]$ is assumed to be associated with heat conductivity parameters $(k_{ij})_\ell$, $\ell = 1, \ldots, n_L$.



In many instances, u exhibits a transient phase which is usually caused by some incompatibilities between the boundary condition and the initial condition. It is shown in [10,1] that, during the transient phase, $\|u_t(t)\|_2 = \|u_t(t)\|_{L_2(\Omega)}$ behaves frequently like $t^{-\alpha}$, $(0 < \alpha < 1)$ as $t \to 0$. Subsequently, a class of nonuniform graded time discretization scheme for a class of linear parabolic equations was established in [1] in order to capture the solution accurately. The result in [1] was extended to nonlinear problems in [3]. We also note that a similar time discretization technique, using a different analysis, was developed in [13,14]. In this paper, we do not make a specific reference to the issue of time discretization, but simply assume that it is done in such a way that optimal convergence rate is attained in time variable. Throughout the paper, it is assumed that the function $a : R \to R^+$ satisfies

$$c \leqslant a(r) \leqslant C, \quad |a'(r)| \leqslant C, \quad r \in R$$

for some positive constants c and C. The weak formulation for (2.1) is given by

Find $u : R^+ \to H_0^1(\Omega)$ such that

$$\begin{aligned} &(u_t(t), v) + (a(u(t))\nabla u(t), \nabla v) = (f(t), v) \quad \text{for all } v \in H_0^1(\Omega), \ t > 0, \\ &u(0) = u_0, \end{aligned} \tag{2.2}$$

where $H_0^1(\Omega)$ is the standard Sobolev space, $(\cdot,\cdot)$ denotes the $L_2(\Omega)$ inner product, and $u_t = \frac{\partial u}{\partial t}$. The DGFEM for approximating the solution of (2.2) using the hierarchical basis function is now described. First, a time interval, $[0, T]$, is partitioned into $0 = t_0 < t_1 < \cdots < t_N = T$. Let $I_n = (t_{n-1}, t_n]$ and $k_n = t_n - t_{n-1}$, $n \geqslant 1$. With q a nonnegative integer, define

$$W \equiv \{v : R^+ \to V_h^r \otimes V : v|_{I_n} \in P_q(I_n), n = 1, \dots N\},$$

where

$$V_h^r = \left\{ \begin{array}{l} \text{the space of splines of order } r \text{ defined over } \omega = \cup_{K\in T_h} K \\ \qquad \text{where } T_h \text{ is a triangulation of } \Omega \text{ and } h = \max\limits_{K\in T_h} \operatorname{diam}(K) \end{array} \right\} = \operatorname{span}\{\varphi_i(x,y)\}_{i=0}^{N_{xy}},$$

$$V = \operatorname{span}\{\psi_j\}_{j=0}^{N_z}.$$

Here $\psi_j \subset H^1(-1,1)$ are linearly independent functions defining a hierarchical modelling

$$P_q(I_n) = \left\{ v(t) = \sum_{\ell=0}^{q} v_\ell \theta_\ell(t) : v_\ell \in V_h^r \otimes V \right\},$$

$$\operatorname{span}\{\theta_\ell(t)\} = \text{the space of all polynomial of degree } \leqslant q \text{ in } t$$

and

$$V_h^r \otimes V = \left\{ \sum_{i=0}^{N_{xy}} \sum_{j=0}^{N_z} \varphi_i(x,y)\psi_j(z) : \varphi_i(x,y) \in V_h^r \quad \text{for each } i \text{ and } \psi_j \in V \text{ for each } j \right\}.$$

Also, define

$$v_n^- = \lim_{s\to t_n^-} v(s), \quad v_n^+ = \lim_{s\to t_n^+} v(s).$$

The DGFEM for (2.2) is given as follows:

Find $U^n \in W$ such that

$$\begin{aligned} &\int_{I_n} \{(U_t^n, v^n) + (a(U^n)\nabla U^n, \nabla v^n)\}\mathrm{d}t + (U_{n-1}^{n+}, v_{n-1}^{n+}) \\ &\quad = (U_{n-1}^{n-}, v_{n-1}^{n+}) + \int_{I_n} (f, v^n)\mathrm{d}t, \quad v^n \in W, \quad n = 1, 2, \dots, N-1, \\ &U^0 = u_0. \end{aligned} \tag{2.3}$$

Furthermore, if we denote $U^{n,K} = U^n|_K$, $K \in T_h$, then the first equation in (2.3) can be written as

$$\sum_{K\in T_h}\int_K\int_{-\frac{d}{2}}^{\frac{d}{2}}\left\{\int_{I_n}\{U_t^{n,K}v^{n,K} + a(U^{n,K})\nabla U^{n,K}\nabla v^{n,K}\}\mathrm{d}t + U_{n-1}^{n,K+}v_{n-1}^{n,K+}\right\}\mathrm{d}z\,\mathrm{d}\omega$$
$$= \sum_{K\in T_h}\int_K\int_{-\frac{d}{2}}^{\frac{d}{2}}\left\{U_{n-1}^{n,K-}v_{n-1}^{n,K+} + \int_{I_n} fv^n\,\mathrm{d}t\right\}\mathrm{d}z\,\mathrm{d}\omega, \quad v^{n,K}\in W, \quad n = 1,2,\ldots,N-1. \tag{2.4}$$

As $U^{n,K} \in W$

$$U^{n,K} = \sum_{i=0}^{N_{xy}}\sum_{j=0}^{N_z}\sum_{k=0}^{q} c_{ijk}^{n,K}\varphi_i(x,y)\psi_j(z)\theta_k(t) \tag{2.5}$$

for some $\{c_{ijk}^{n,K}\}$. Writing (2.5) as

$$U^{n,K} = \chi^{\mathrm{T}}\mathbf{c}^{n,K} = (\varphi\otimes\psi\otimes\theta)^{\mathrm{T}}\mathbf{c}^{n,K} \tag{2.6}$$

and letting $v^{n,K} = \chi$ in (2.4)

$$\sum_{K\in T_h}\int_K\int_{-\frac{d}{2}}^{\frac{d}{2}}\left\{\int_{I_n}\{\chi\frac{\partial\chi^{\mathrm{T}}}{\partial t}\mathbf{c}^{n,K} + a(\chi\mathbf{c}^{n,K})(\nabla\chi^{\mathrm{T}})^{\mathrm{T}}\nabla\chi^{\mathrm{T}}\mathbf{c}^{n,K}\}\mathrm{d}t + \chi^+(\chi^+)^{\mathrm{T}}\mathbf{c}^{n,K}\right\}\mathrm{d}z\,\mathrm{d}\omega$$
$$= \sum_{K\in T_h}\int_K\int_{-\frac{d}{2}}^{\frac{d}{2}}\left\{\chi^-(\chi^+)^{\mathrm{T}}\mathbf{c}^{n-1,K} + \int_{I_n} f\chi\,\mathrm{d}t\right\}\mathrm{d}z\,\mathrm{d}\omega, \quad \chi\in W, \quad n = 1,2,\ldots,N-1. \tag{2.7}$$

Note that $\mathbf{c}^{0,K}$ is determined by the initial condition $u_0(x,y,z)$. The element matrices and load vectors in (2.7) are defined as

$$\begin{aligned}
[C_K] &= \int_K\int_{-\frac{d}{2}}^{\frac{d}{2}}\int_{I_n}\chi\frac{\partial\chi^{\mathrm{T}}}{\partial t}\mathrm{d}t\,\mathrm{d}z\,\mathrm{d}\omega,\\
[A(\chi\mathbf{c}^{n,K})] &= \int_K\int_{-\frac{d}{2}}^{\frac{d}{2}}\int_{I_n} a(\chi^{\mathrm{T}}\mathbf{c}^{n,K}))(\nabla\chi^{\mathrm{T}})^{\mathrm{T}}\nabla\chi^{\mathrm{T}}\,\mathrm{d}t\,\mathrm{d}z\,\mathrm{d}\omega,\\
[M_K^{++}] &= \int_K\int_{-\frac{d}{2}}^{\frac{d}{2}}\chi^+(\chi^+)^{\mathrm{T}}\mathrm{d}z\,\mathrm{d}\omega,\\
[M_K^{-+}] &= \int_K\int_{-\frac{d}{2}}^{\frac{d}{2}}\chi^-(\chi^+)^{\mathrm{T}}\mathrm{d}z\,\mathrm{d}\omega,\\
\{H_K\} &= \int_K\int_{-\frac{d}{2}}^{\frac{d}{2}}\int_{I_n} f\chi\,\mathrm{d}t\,\mathrm{d}z\,\mathrm{d}\omega,
\end{aligned} \tag{2.8}$$

where $[C_K]$ represents the element capacitance matrix, $[A(\chi)]$ represents the element conductance matrix, $[M_K^{++}]$ and $[M_K^{-+}]$ represent element mass matrices, and $\{H_K\}$ is the element load vector. Using these notations, (2.7) now becomes

$$\sum_{K\in T_h}\left[[C_K] + [A(\chi^{\mathrm{T}}\mathbf{c}^{n,K})] + [M_K^{++}]\right]\{\mathbf{c}^{n,K}\} = \sum_{K\in T_h}\left\{[M_K^{-+}]\{\mathbf{c}^{n-1,K}\} + \{H_K\}\right\}. \tag{2.9}$$

An interesting variation of (2.9) is

$$\sum_{K\in T_h}\left[[C_K] + [A(\chi^{\mathrm{T}}\mathbf{c}^{n-1,K})] + [M_K^{++}]\right]\{\tilde{\mathbf{c}}^{n,K}\} = \sum_{K\in T_h}\left\{[M_K^{-+}]\{\mathbf{c}^{n-1,K}\} + \{H_K\}\right\}. \tag{2.10}$$

Here, of course, the unknown $\mathbf{c}^{n,K}$ that appears under nonlinear term A in (2.9) is replaced by $\mathbf{c}^{n-1,K}$. Eq. (2.10) is now linear in $\tilde{\mathbf{c}}^{n,K}$. Denote respectively (2.9) and (2.10) more concisely as

$$[S(\mathbf{c}^{n,K})]\{\mathbf{c}^{n,K}\} = [F(\mathbf{c}^{n-1,K})], \tag{2.11}$$
$$[S(\mathbf{c}^{n-1,K})]\{\tilde{\mathbf{c}}^{n,K}\} = [F(\mathbf{c}^{n-1,K})]. \tag{2.12}$$

Eq. (2.12) suggests the following iteration scheme that can be used to approximate the solution $\mathbf{c}^{n,K}$ of the non-linear equation (2.11). First, let $\mathbf{c}_0 = \mathbf{c}^{n-1,K}$ for each $n \geqslant 1$ and define $\mathbf{c}_1$ from

$$[S(\mathbf{c}_0)]\{\mathbf{c}_1\} = [F(\mathbf{c}^{n-1,K})].$$

Inductively, find $\mathbf{c}_k$ from

$$[S(\mathbf{c}_{k-1})]\{\mathbf{c}_k\} = [F(\mathbf{c}^{n-1,K})]$$

or more specifically

$$\sum_{K\in T_h}\left[[C_K] + [A(\chi^{\mathrm{T}}\mathbf{c}_{k-1})] + \left[M_K^{++}\right]\right]\{\mathbf{c}_k\} = \sum_{K\in T_h}\left\{\left[M_K^{-+}\right]\{\mathbf{c}^{n-1,K}\} + \{H_K\}\right\}. \tag{2.13}$$

To see the convergence of $\mathbf{c}_k$ to $\mathbf{c}^{n,K}$ as $k \to \infty$ under some appropriate conditions, first note that

$$\begin{aligned}\mathbf{c}^{n,K} - \mathbf{c}_k &= [S(\mathbf{c}^{n-1,K})]^{-1}[F(\mathbf{c}^{n-1,K})] - [S(\mathbf{c}_{k-1})]^{-1}[F(\mathbf{c}^{n-1,K})]\\ &= [S(\mathbf{c}^{n-1,K})]^{-1}\{[S(\mathbf{c}_{k-1})] - [S(\mathbf{c}^{n-1,K})]\}[S(\mathbf{c}_{k-1})]^{-1}[F(\mathbf{c}^{n-1,K})].\end{aligned} \tag{2.14}$$

Since $[S(\cdot)] : R^D \to R^{D\times D}$, $D \equiv (N_{xy}+1)\times(N_z+1)\times(q+1)$, we see that with $[S(\cdot)] = [s_{ij}(\cdot)]_{i,j=1}^D$, $s_{ij} : R^D \to R$, for each i and j

$$[S(\mathbf{c}_{k-1})] - [S(\mathbf{c}^{n-1,K})] = \begin{bmatrix} \nabla s_{11}(\bar{\eta}_{11})\cdot(\mathbf{c}_{k-1} - \mathbf{c}^{n-1,K}) & \cdots & \nabla s_{1D}(\bar{\eta}_{1D})\cdot(\mathbf{c}_{k-1} - \mathbf{c}^{n-1,K}) \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ \nabla s_{D1}(\bar{\eta}_{D1})\cdot(\mathbf{c}_{k-1} - \mathbf{c}^{n-1,K}) & \cdots & \nabla s_{DD}(\bar{\eta}_{DD})\cdot(\mathbf{c}_{k-1} - \mathbf{c}^{n-1,K}) \end{bmatrix},$$

where $\bar{\eta}_{ij}$ is between $\mathbf{c}_{k-1}$ and $\mathbf{c}^{n-1,K}$. Hence, assuming

$$\max_{1\leqslant i\leqslant D}\sum_{j=1}^{D}\|\nabla s_{ij}(\bar{\eta}_{ij})\|_2 \leqslant C_1 \quad \text{for some } C_1 > 0, \tag{2.15}$$

$$\begin{aligned}\|[S(\mathbf{c}_{k-1})] - [S(\mathbf{c}^{n-1,K})]\|_\infty &= \max_{1\leqslant i\leqslant D}\sum_{j=1}^{D}|\nabla s_{ij}(\bar{\eta}_{ij})\cdot[\mathbf{c}_{k-1} - \mathbf{c}^{n-1,K}]|\\ &\leqslant \max_{1\leqslant i\leqslant D}\sum_{j=1}^{D}\|\nabla s_{ij}(\bar{\eta}_{ij})\|_2\|\mathbf{c}_{k-1} - \mathbf{c}^{n-1,K}\|_2\\ &\leqslant C_1\sqrt{D}\|\mathbf{c}_{k-1} - \mathbf{c}^{n-1,K}\|_\infty.\end{aligned} \tag{2.16}$$

Furthermore, under the assumptions that

$$\|[S(\mathbf{c}^{n-1,K})]^{-1}\|_\infty \leqslant C_2, \quad \|[S(\mathbf{c}_k)]^{-1}\|_\infty \leqslant C_2, \quad \|[F(\mathbf{c}^{n-1,K})]\|_\infty \leqslant C_2 \tag{2.17}$$

for each n and k and for some $C_2 > 0$, (2.14)–(2.17) imply that

$$\|\mathbf{c}^{n,K} - \mathbf{c}_k\|_\infty \leqslant C_1C_2\sqrt{D}\|\mathbf{c}^{n,K} - \mathbf{c}_{k-1}\|_\infty. \tag{2.18}$$

If $H \equiv D_1D_2\sqrt{D}$ in (2.18), then provided that $0 < H < 1$

$$\|\mathbf{c}^{n,K} - \mathbf{c}_k\|_\infty \leqslant H^k\|\mathbf{c}^{n,K} - \mathbf{c}_0\|_\infty \to 0$$

as $k \to \infty$. Hence, we have now proved the following:

Theorem 2.1. *Let $\mathbf{c}^{n,K}$ be a unique solution of (2.9), equivalently (2.11), and let $\mathbf{c}_k$ be defined by (2.13) with $\mathbf{c}_0 = \mathbf{c}^{n-1,K}$ for each n. If the constants C_1 and C_2 defined respectively by (2.15) and (2.17) satisfy $C_1C_2\sqrt{D} < 1$, then $\mathbf{c}_k$ converges to $\mathbf{c}^{n,K}$ as $k \to \infty$ for each n.*

Please cite this article in press as: H. Kaneko et al., Numerical experiments using hierarchical finite element ..., Appl. Math. Comput. (2008), doi:10.1016/j.amc.2007.12.062

If $C_1C_2\sqrt{D} < 1$ is not satisfied, then one should modify (2.12) by multiplying both sides by a pre-conditioner M^{-1}, i.e.,

$$M^{-1}[S(\mathbf{c}^{n-1,K})]\{\mathbf{c}^{n,K}\} = M^{-1}[F(\mathbf{c}^{n-1,K})],$$

so that the bound $C_1C_2\sqrt{D} < 1$ is satisfied relative to $[S(\mathbf{c}^{n-1,K})]^{-1}M$, $[S(\mathbf{c}_k)]^{-1}M$, and $M^{-1}[F(\mathbf{c}^{n-1,K})]$. One may select, for instance, M to be the diagonal matrix which contains $\|[S(\mathbf{c}^{n-1,K})]\|_\infty$ along its diagonal.

The element matrices defined in (2.8) may be reduced to more convenient forms. Capacitance matrix, for example, can be given as follows:

$$\begin{aligned}[C_K] &= \int_K \int_{-\frac{d}{2}}^{\frac{d}{2}} \int_{I_n} \chi \frac{\partial \chi^{\mathrm{T}}}{\partial t}\,\mathrm{d}t\,\mathrm{d}z\,\mathrm{d}\omega = \int_K \int_{-\frac{d}{2}}^{\frac{d}{2}} \int_{I_n} (\varphi\otimes\psi\otimes\theta)\left(\varphi\otimes\psi\otimes\frac{\mathrm{d}\theta}{\mathrm{d}t}\right)^{\mathrm{T}}\mathrm{d}t\,\mathrm{d}z\,\mathrm{d}\omega \\ &= \int_K \varphi\varphi^{\mathrm{T}}\,\mathrm{d}\omega \otimes \int_{-\frac{d}{2}}^{\frac{d}{2}} \psi\psi^{\mathrm{T}}\,\mathrm{d}z \otimes \int_{-\frac{d}{2}}^{\frac{d}{2}} \theta\frac{\mathrm{d}\theta^{\mathrm{T}}}{\mathrm{d}t}\,\mathrm{d}t.\end{aligned}$$

More specifically, with respect to the following three matrices $[\Phi] \in R^{(N_{xy}+1)\times(N_{xy}+1)}$, $[\Psi] \in R^{(N_z+1)\times(N_z+1)}$ and $\left[\Theta\frac{\mathrm{d}\Theta^{\mathrm{T}}}{\mathrm{d}t}\right] \in R^{(q+1)\times(q+1)}$ defined by

$$\begin{aligned}[\Phi] &= \int_K \varphi\varphi^{\mathrm{T}}\,\mathrm{d}\omega, \\ [\Psi] &= \int_{-\frac{d}{2}}^{\frac{d}{2}} \psi\psi^{\mathrm{T}}\,\mathrm{d}z, \\ \left[\Theta\frac{\mathrm{d}\Theta^{\mathrm{T}}}{\mathrm{d}t}\right] &= \int_{I_n} \theta\frac{\mathrm{d}\theta^{\mathrm{T}}}{\mathrm{d}t}\,\mathrm{d}t,\end{aligned}$$

the capacitance matrix is obtained from

$$[C_K] = [\Phi]\otimes[\Psi]\otimes\left[\Theta\frac{\mathrm{d}\Theta^{\mathrm{T}}}{\mathrm{d}t}\right]. \tag{2.19}$$

The operation of the outer tensor $\otimes$ between matrices is defined in the standard way which, for completeness, is illustrated below by a 2×2 matrix A and a 3×3 matrix B

$$[A] = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \quad [B] = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix},$$

$$[A]\otimes[B] = \begin{bmatrix} a_{11}[B] & a_{12}[B] \\ a_{21}[B] & a_{22}[B] \end{bmatrix} = \begin{bmatrix} a_{11}b_{11} & a_{11}b_{12} & a_{11}b_{13} & a_{12}b_{11} & a_{12}b_{12} & a_{12}b_{13} \\ a_{11}b_{21} & a_{11}b_{22} & a_{11}b_{23} & a_{12}b_{21} & a_{12}b_{22} & a_{12}b_{23} \\ a_{11}b_{31} & a_{11}b_{32} & a_{11}b_{33} & a_{12}b_{31} & a_{12}b_{32} & a_{12}b_{33} \\ a_{21}b_{11} & a_{21}b_{12} & a_{21}b_{13} & a_{22}b_{11} & a_{22}b_{12} & a_{22}b_{13} \\ a_{21}b_{21} & a_{21}b_{22} & a_{21}b_{23} & a_{22}b_{21} & a_{22}b_{22} & a_{22}b_{23} \\ a_{21}b_{31} & a_{21}b_{32} & a_{21}b_{33} & a_{22}b_{31} & a_{22}b_{22} & a_{22}b_{33} \end{bmatrix}.$$

Similarly to (2.11), $[M_K^{++}]$, $[M_K^{-+}]$ and $\{H_K\}$ in (2.8) are given by

$$\begin{aligned}\left[M_K^{++}\right] &= [\Phi]\otimes[\Psi]\otimes\Theta^{+}(\Theta^{+})^{\mathrm{T}}, \\ \left[M_K^{-+}\right] &= [\Phi]\otimes[\Psi]\otimes\Theta^{-}(\Theta^{+})^{\mathrm{T}}, \\ \{H_K\} &= \int_K \int_{-\frac{d}{2}}^{\frac{d}{2}} \int_{I_n} f(\cdot)\cdot(\varphi\otimes\psi\otimes\theta)\mathrm{d}t\,\mathrm{d}z\,\mathrm{d}\omega,\end{aligned} \tag{2.20}$$

In order to efficiently solve (2.13) for each k, it is important to preassemble as many of the element matrices in (2.10) as possible. Therefore, besides (2.19) and (2.20), it remains to consider the integral:

$$\int_K \int_{-\frac{d}{2}}^{\frac{d}{2}} \int_{I_n} a(\chi^{\mathrm{T}}\mathbf{c}_{k-1})(\nabla\chi^{\mathrm{T}})^{\mathrm{T}}\nabla\chi^{\mathrm{T}}\,\mathrm{d}t\,\mathrm{d}z\,\mathrm{d}\omega. \tag{2.21}$$

If $a(\cdot) \equiv a$ is a constant function, then for each k

$$[A(\chi^{\mathrm{T}}\mathbf{c}_{k-1})] = a\{[\Phi_x]\otimes[\Psi]\otimes[\Theta] + [\Phi_y]\otimes[\Psi]\otimes[\Theta] + [\Phi]\otimes[\Psi_z]\otimes[\Theta]\},$$

where

$$[\Phi_x] = \int_K \varphi_x\varphi_x^{\mathrm{T}}\,\mathrm{d}\omega,$$
$$[\Phi_y] = \int_K \varphi_y\varphi_y^{\mathrm{T}}\,\mathrm{d}\omega,$$
$$[\Psi_z] = \int_{-\frac{d}{2}}^{\frac{d}{2}} \psi_z\psi_z^{\mathrm{T}}\,\mathrm{d}z.$$

For a more general function a, integral (2.21) must be computed at each step k of iteration. To reduce the computational cost, we proceed as follows: As $a(\chi^{\mathrm{T}}\mathbf{c}_{k-1})$ is known for each k in (2.12), one may approximate it by its L_2 projection $\chi^{\mathrm{T}}\mathbf{b}_{k-1}$. For instance, if we take the space spanned by $\{\chi\}$ to be the projected space, then $\mathbf{b}_{k-1}$ is computed from the fact that $a(\chi^{\mathrm{T}}\mathbf{c}_{k-1}) - \chi^{\mathrm{T}}\mathbf{b}_{k-1}$ is orthogonal to each χ, i.e.,

$$\int_K \int_{-\frac{d}{2}}^{\frac{d}{2}} \int_{I_n} \chi\chi^{\mathrm{T}}\,\mathrm{d}t\,\mathrm{d}z\,\mathrm{d}\omega\{\mathbf{b}_{k-1}\} = \int_K \int_{-\frac{d}{2}}^{\frac{d}{2}} \int_{I_n} a(\chi^{\mathrm{T}}\mathbf{c}_{k-1})\chi\,\mathrm{d}t\,\mathrm{d}z\,\mathrm{d}\omega \tag{2.22}$$

or in the matrix form

$$[[\Phi]\otimes[\Psi]\otimes[\Theta]]\{\mathbf{b}_{k-1}\} = \{\mathbf{A}\}_{k-1}, \tag{2.23}$$

where $\{\mathbf{A}\}_{k-1} = \int_K \int_{-\frac{d}{2}}^{\frac{d}{2}} \int_{I_n} a(\chi^{\mathrm{T}}\mathbf{c}_{k-1})\chi\,\mathrm{d}t\,\mathrm{d}z\,\mathrm{d}\omega$. By replacing $a(\chi^{\mathrm{T}}\mathbf{c}_{k-1})$ by $\chi^{\mathrm{T}}\mathbf{b}_{k-1}$, (2.21) now becomes

$$\int_K \int_{-\frac{d}{2}}^{\frac{d}{2}} \int_{I_n} \chi^{\mathrm{T}}\mathbf{b}_{k-1}(\nabla\chi^{\mathrm{T}})^{\mathrm{T}}\nabla\chi^{\mathrm{T}}\,\mathrm{d}t\,\mathrm{d}z\,\mathrm{d}\omega. \tag{2.24}$$

Since $\chi^{\mathrm{T}}\mathbf{b}_{k-1}$ is the L_2 projection of $a(\chi^{\mathrm{T}}\mathbf{c}_{k-1})$ onto the space spanned by $\{\chi\}$, using the technique established in [2] to derive an error bound for a modified DGFEM in which h-finite element approximation was used for the surface of a plate whereas p-finite element approximation was used through the thickness, it can be seen that

$$\|a(\chi^{\mathrm{T}}\mathbf{c}_{k-1}) - \chi^{\mathrm{T}}\mathbf{b}_{k-1}\|_2 = \mathrm{O}(h^{N+1} + d^{N_z} + I_n^{q+1}). \tag{2.25}$$

The second term $O\left(\left(\frac{1}{N_z}\right)^m\right)$ in (2.25) is the error term associated with the hierarchical modelling used through the thickness of a plate and a more complete discussion of this error term will be given in the next section. It should be pointed out that the modified DGFEM of [2] deals with linear parabolic equations and also it is not based upon the hierarchical modelling technique discussed in this paper.

Dropping for simplicity the index $k-1$ from the components of $\mathbf{b}_{k-1}$, we write $\mathbf{b}_{k-1} = (b_\ell)_{\ell=1}^D$. Also, let $\chi = \{\chi_\ell\}_\ell^D = \{\varphi_{\ell_i}\psi_{\ell_j}\theta_{\ell_k}\}_{i=1,j=1,k=1}^{N+1,N_z+1,q+1}$. Then

$$\begin{aligned}\chi^{\mathrm{T}}\mathbf{b}_{k-1}(\nabla\chi^{\mathrm{T}})^{\mathrm{T}}\nabla\chi^{\mathrm{T}} &= \sum_{\ell=1}^{D}\chi_\ell b_\ell\left[\left[\frac{\partial\varphi}{\partial x}\frac{\partial\varphi^{\mathrm{T}}}{\partial x} + \frac{\partial\varphi}{\partial y}\frac{\partial\varphi^{\mathrm{T}}}{\partial y}\right]\otimes\psi\psi^{\mathrm{T}}\otimes\theta\theta^{\mathrm{T}} + \varphi\varphi^{\mathrm{T}}\otimes\left[\frac{\partial\psi}{\partial z}\frac{\partial\psi^{\mathrm{T}}}{\partial z}\right]\otimes\theta\theta^{\mathrm{T}}\right]\\ &= \sum_i \varphi_{\ell_i}b_{\ell_i}\left[\frac{\partial\varphi}{\partial x}\frac{\partial\varphi^{\mathrm{T}}}{\partial x} + \frac{\partial\varphi}{\partial y}\frac{\partial\varphi^{\mathrm{T}}}{\partial y}\right]\otimes\sum_j \psi_{\ell_j}b_{\ell_j}\psi\psi^{\mathrm{T}}\otimes\sum_k \theta_{\ell_k}b_{\ell_k}\theta\theta^{\mathrm{T}}\\ &\quad + \sum_i \varphi_{\ell_i}b_{\ell_i}\varphi\varphi^{\mathrm{T}}\otimes\sum_j \psi_{\ell_j}b_{\ell_j}\left[\frac{\partial\psi}{\partial z}\frac{\partial\psi^{\mathrm{T}}}{\partial z}\right]\otimes\sum_k \theta_{\ell_k}b_{\ell_k}\theta\theta^{\mathrm{T}}.\end{aligned} \tag{2.26}$$

Substituting the second expression of (2.26) into (2.24), one obtains

$$
\begin{aligned}
\int_K \int_{-\frac{d}{2}}^{\frac{d}{2}} \int_{I_n} \chi^{\mathrm{T}} \mathbf{b}_{k-1} (\nabla \chi^{\mathrm{T}})^{\mathrm{T}} \nabla \chi^{\mathrm{T}} \,\mathrm{d}t\,\mathrm{d}z\,\mathrm{d}\omega &= \sum_i b_\ell \int_K \varphi_{\ell_i} \left[\frac{\partial \varphi}{\partial x} \frac{\partial \varphi^{\mathrm{T}}}{\partial x} + \frac{\partial \varphi}{\partial y} \frac{\partial \varphi^{\mathrm{T}}}{\partial y} \right] \mathrm{d}K \otimes \sum_j b_{\ell_j} \\
&\quad \times \int_{-\frac{d}{2}}^{\frac{d}{2}} \psi_{\ell_j} \psi \psi^{\mathrm{T}} \,\mathrm{d}z \otimes \sum_k b_{\ell_k} \int_{I_n} \theta_{\ell_k} \theta \theta^{\mathrm{T}} \,\mathrm{d}t + \sum_i b_{\ell_i} \\
&\quad \times \int_K \varphi_{\ell_i} \varphi \varphi^{\mathrm{T}} \,\mathrm{d}K \otimes \sum_j b_{\ell_j} \int_{-\frac{d}{2}}^{\frac{d}{2}} \psi_{\ell_j} \left[\frac{\partial \psi}{\partial z} \frac{\partial \psi^{\mathrm{T}}}{\partial z} \right] \mathrm{d}z \otimes \sum_k b_{\ell_k} \\
&\quad \times \int_{I_n} \theta_{\ell_k} \theta \theta^{\mathrm{T}} \,\mathrm{d}t \\
&\equiv [B(\chi^{\mathrm{T}} \mathbf{b}_{k-1})].
\end{aligned} \tag{2.27}
$$

All the integrals in (2.27) should be preassembled along with (2.19) and (2.20). The solution $\mathbf{c}_k$ of (2.13) is now approximated by solving for $\tilde{\mathbf{c}}_k$ of the following:

$$
\sum_{K \in T_h} \left[[C_K] + [B(\chi^{\mathrm{T}} \mathbf{b}_{k-1})] + \left[M_K^{++} \right] \right] \{ \tilde{\mathbf{c}}_k \} = \sum_{K \in T_h} \left\{ \left[M_K^{-+} \right] \left\{ \mathbf{c}^{n-1,K} \right\} + \{ H_K \} \right\}. \tag{2.28}
$$

Analogous to (2.12), we write (2.27) more concisely as

$$
[\widetilde{S}(\mathbf{b}_{k-1})] \{ \tilde{\mathbf{c}}_k \} = [F(\mathbf{c}^{n-1,K})]. \tag{2.29}
$$

To see that $\|\mathbf{c}_k - \tilde{\mathbf{c}}_k\|_\infty \to 0$ as $N, N_z, q \to \infty$

$$
\begin{aligned}
\|\mathbf{c}_k - \tilde{\mathbf{c}}_k\|_\infty &= \| [S(\mathbf{c}_{k-1})^{-1} - \widetilde{S}(\mathbf{b}_{k-1})^{-1}] [F(\mathbf{c}^{n-1,K})] \|_\infty \\
&= \| S(\mathbf{c}_{k-1})^{-1} [\widetilde{S}(\mathbf{b}_{k-1}) - S(\mathbf{c}_{k-1})] \widetilde{S}(\mathbf{b}_{k-1})^{-1} [F(\mathbf{c}^{n-1,K})] \|_\infty \leqslant C_3 \| \widetilde{S}(\mathbf{b}_{k-1}) - S(\mathbf{c}_{k-1}) \|_\infty,
\end{aligned} \tag{2.30}
$$

where $C_3 > 0$ is a bound for $\|S(\mathbf{c}_{k-1})^{-1}\|_\infty \|\widetilde{S}(\mathbf{b}_{k-1})^{-1}\|_\infty \|[F(\mathbf{c}^{n-1,K})]\|_\infty$. Now

$$
\begin{aligned}
\|\widetilde{S}(\mathbf{b}_{k-1}) - S(\mathbf{c}_{k-1})\|_\infty &= \left\| \sum_{K \in T_h} [B(\chi^{\mathrm{T}} \mathbf{b}_{k-1}) - A(\chi^{\mathrm{T}} \mathbf{c}_{k-1})] \right\|_\infty \\
&= \left\| \sum_{K \in T_h} \int_K \int_{-\frac{d}{2}}^{\frac{d}{2}} \int_{I_n} \{ \chi^{\mathrm{T}} \mathbf{b}_{k-1} - a(\chi^{\mathrm{T}} \mathbf{c}_{k-1}) \} (\nabla \chi^{\mathrm{T}})^{\mathrm{T}} \nabla \chi^{\mathrm{T}} \,\mathrm{d}t\,\mathrm{d}z\,\mathrm{d}\omega \right\|_\infty \\
&= \max_{1 \leqslant i \leqslant D} \sum_{j=1}^{D} \left| \left[\sum_{K \in T_h} \int_K \int_{-\frac{d}{2}}^{\frac{d}{2}} \int_{I_n} \{ \chi^{\mathrm{T}} \mathbf{b}_{k-1} - a(\chi^{\mathrm{T}} \mathbf{c}_{k-1}) \} (\nabla \chi^{\mathrm{T}})^{\mathrm{T}} \nabla \chi^{\mathrm{T}} \,\mathrm{d}t\,\mathrm{d}z\,\mathrm{d}\omega \right]_{ij} \right| \\
&\leqslant \max_{1 \leqslant i \leqslant D} \sum_{j=1}^{D} \sum_{K \in T_h} \left\{ \int_K \int_{-\frac{d}{2}}^{\frac{d}{2}} \int_{I_n} |\chi^{\mathrm{T}} \mathbf{b}_{k-1} - a(\chi^{\mathrm{T}} \mathbf{c}_{k-1})|^2 \mathrm{d}t\,\mathrm{d}z\,\mathrm{d}\omega \right\}^{1/2} \\
&\quad \times \left\{ \int_K \int_{-\frac{d}{2}}^{\frac{d}{2}} \int_{I_n} \left| \left[(\nabla \chi^{\mathrm{T}})^{\mathrm{T}} \nabla \chi^{\mathrm{T}} \right]_{ij} \right|^2 \mathrm{d}t\,\mathrm{d}z\,\mathrm{d}\omega \right\}^{1/2} \\
&\leqslant \max_{1 \leqslant i \leqslant D} \sum_{j=1}^{D} \sum_{K \in T_h} \left\{ \int_K \int_{-\frac{d}{2}}^{\frac{d}{2}} \int_{I_n} \left| \left[(\nabla \chi^{\mathrm{T}})^{\mathrm{T}} \nabla \chi^{\mathrm{T}} \right]_{ij} \right|^2 \mathrm{d}t\,\mathrm{d}z\,\mathrm{d}\omega \right\}^{1/2} \\
&\quad \times \|\chi^{\mathrm{T}} \mathbf{b}_{k-1} - a(\chi^{\mathrm{T}} \mathbf{c}_{k-1})\|_2 \\
&\leqslant C_4 \|\chi^{\mathrm{T}} \mathbf{b}_{k-1} - a(\chi^{\mathrm{T}} \mathbf{c}_{k-1})\|_2,
\end{aligned} \tag{2.31}
$$

where $D_4 = \max_{1 \leqslant i \leqslant D} \sum_{j=1}^{D} \sum_{K \in T_h} \{ \int_K \int_{-\frac{d}{2}}^{\frac{d}{2}} \int_{I_n} |[(\nabla \chi^{\mathrm{T}})^{\mathrm{T}} \nabla \chi^{\mathrm{T}}]_{ij}|^2 \mathrm{d}t\,\mathrm{d}z\,\mathrm{d}\omega \}^{1/2}$. Putting together (2.25), (2.31) and (2.25) we obtain

$$
\|\mathbf{c}_k - \tilde{\mathbf{c}}_k\|_\infty \leqslant C_3 C_4 \|\chi^{\mathrm{T}} \mathbf{b}_{k-1} - a(\chi^{\mathrm{T}} \mathbf{c}_{k-1})\|_2 \to 0, \quad \text{as } N, N_z, q \to \infty. \tag{2.32}
$$

Since $\mathbf{c}_k \to \mathbf{b}^{n,K}$ as $k \to \infty$ by Theorem 2.1, using (2.32)

$$\|\mathbf{b}^{n,K} - \tilde{\mathbf{c}}_k\|_\infty \leqslant \|\mathbf{b}^{n,K} - \mathbf{c}_k\|_\infty + \|\mathbf{c}_k - \tilde{\mathbf{c}}_k\|_\infty \to 0, \quad \text{as } N, N_z, q \to \infty.$$

Thus we proved the following:

Theorem 2.2. *Let $\mathbf{c}^{n,K}$ be a unique solution of (2.9), equivalently (2.11), and let $\mathbf{c}_k$ be defined by (2.13) with $\mathbf{c}_0 = \mathbf{c}^{n-1,K}$ for each n. Assume that the constants C_1 and C_2 defined, respectively, by (2.15) and (2.17) satisfy $C_1C_2\sqrt{D} < 1$. Also let $\tilde{\mathbf{c}}_k$ be the solution of (2.29), then $\tilde{\mathbf{c}}_k$ converges to $\mathbf{c}^{n,K}$ as $k \to \infty$ for each n.*

Finally, in this section, we examine the accuracy of DGFEM solution U, i.e., the solution of U of (2.4), as an approximation of the solution u of (2.1). For this end, an important theorem by Erikkson and Johnson is recalled. As was done in [4], the following will be assumed. These assumptions allow u to have a transient phase. More importantly, these assumptions form the basis for Theorem 2.3 below:

$$\max_{t \leqslant t_N} \|u(t)\|_\infty \leqslant C_5, \tag{2.33}$$

$$\int_0^{t_N} \|\nabla u\|_\infty^2 \leqslant C_6, \tag{2.34}$$

$$\sum_{n=1}^{N} k_n \|\nabla u\|_\infty^2 \leqslant C_7, \tag{2.35}$$

$$\max_{M \leqslant N} \left(\log \frac{t_M}{k_M} + 1\right) + \sum_{n=1}^{M} k_n \left\| a(u_n)\nabla u_n - \frac{1}{k_n}\int_{I_n} a(u)\nabla u \, \mathrm{d}t \right\|_\infty^2 \leqslant C_8, \tag{2.36}$$

$$\int_0^{t_N} \|u_t(t)\|_\infty \mathrm{d}t \leqslant C_9, \tag{2.37}$$

$$\|u_t(t)\|_2 \leqslant C_{10} t^{-1+\beta}, \tag{2.38}$$

where $C_1 - C_{10}$ and β are positive constants.

The following theorem is given in [4] for $q = 0$. The case for $q = 1$, the linear in time, is treated in [6].

Theorem 2.3 [4]. *Let u be the solution of (2.1) and $U \in W$ that of (2.4) and assume that (2.33)–(2.36) hold. Then there is a constant C depending only on the bounds for a, a', $C_5 - C_8$, and μ with $k_{n-1} \leqslant \mu k_n$ for all $1 < n \leqslant N$, such that*

$$\max_{t \leqslant t_N} \|u(t) - U(t)\|_2 \leqslant C\left(\log \frac{t_N}{k_N} + 1\right)^{1/2} \inf_{v \in W} \max_{t \leqslant t_N} \|u(t) - v(t)\|_2. \tag{2.39}$$

Putting together (2.39) in Theorem 2.3 and (2.25), we obtain

Theorem 2.4. *Let u be the solution of (2.1) and $U \in W$ that of (2.4) and assume that (2.33)–(2.36) hold. Then there is a constant C depending only on the bounds for a, a', $C_5 - C_8$, and μ with $k_{n-1} \leqslant \mu k_n$ for all $1 < n \leqslant N$, such that*

$$\max_{t \leqslant t_N} \|u(t) - U(t)\|_2 \leqslant C\left(\log \frac{t_N}{k_N} + 1\right)^{1/2} (h^{N+1} + d^{N_z} + I_n^{q+1}). \tag{2.40}$$

3. Hierarchical modelling and numerical examples

Hierarchical modelling is a method of approximating a solution of boundary value problems (particularly of elliptic type) on domains which has a thin structure in at least one transverse direction, such as plates and shells. This method has been used extensively in various engineering applications. The authors are not aware of a study which incorporates the idea of hierarchical modelling in solving parabolic problems as was done in this paper.

A class of hierarchical functions which produce an optimal convergence result is always problem dependent. For example, for a steady-state heat equation below proposed on $\Omega = \omega \times (-d, d) \subset R^{n+1}$, $n \geqslant 2$, $-1 < d < 1$ with the sets $R_{\pm} = \omega \times \{\pm d\}$, $\Gamma = \gamma \times (-d, d)$

$$\begin{aligned} &\Delta u = 0 \quad \text{in } \Omega, \\ &u = 0 \quad \text{on } \Gamma, \\ &\frac{\partial u}{\partial n} = f^{\pm} \quad \text{on } R_{\pm}. \end{aligned} \tag{3.1}$$

Vogelius and Babuska [15,16] showed that an optimal choice of hierarchical basis functions for steady-state problem (3.1) can be found as follows: Define $\psi_{2j}(z) = \psi_{2j}(-z)$, $j = 0, 1, \ldots,$ recursively

$$\int_{-1}^{1} a(z)\psi_0'(z)v'(z)\mathrm{d}z = 0, \tag{3.2}$$

$$\int_{-1}^{1} a(z)\psi_{2j}'(z)v'(z)\mathrm{d}z + \int_{-1}^{1} b(z)\psi_{2j-2}(z)v(z)\mathrm{d}z = \delta_j(v) \tag{3.3}$$

for all $v \in H^1[-1, 1]$, $j = 1, 2, \ldots$, where

$$\delta_j(v) = \begin{cases} v(1) + v(-1) & \text{if } j = 1, \\ 0 & \text{else} \end{cases}$$

and define $\psi_{2j+1}(z) = -\psi_{2j+1}(-z)$, $j = 0, 1, \ldots$ by

$$\int_{-1}^{1} a(z)\psi_{2j+1}'(z)v'(z)\mathrm{d}z + \int_{-1}^{1} b(z)\psi_{2j-1}(z)v(z)\mathrm{d}z = \tilde{\delta}_j(v) \tag{3.4}$$

for all $v \in H^1[-1, 1]$, $j = 1, 2, \ldots$, where

$$\tilde{\delta}_j(v) = \begin{cases} v(1) - v(-1) & \text{if } j = 0, \\ 0 & \text{else} \end{cases}$$

and

$$\psi_{-1} = 0. \tag{3.5}$$

For nonlinear parabolic problems, the optimal choice of hierarchical basis functions is difficult to determine. We demonstrate below that a collection of basis functions $\{z^{2i}\}_{i=0}^{n}$ works quite well for model examples considered in this paper.

Example 1. This example examines the effectiveness of the current method of linearization applied to (2.1) over two-dimensional plates. Here we assume that $\Omega = [-2, 2] \times [-2, 2]$, $[0, T] = [0, 5]$ and the solution

$$u(x, y, t) = (1 - 0.25x^2)(1 - 0.25y^2)t,$$

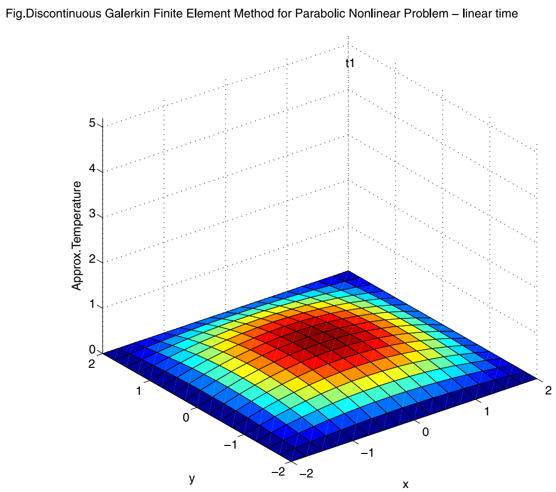
where

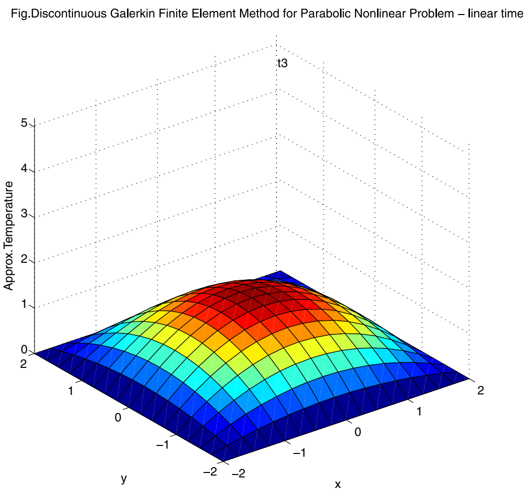
$$k(u) = 0.005u + 1.$$

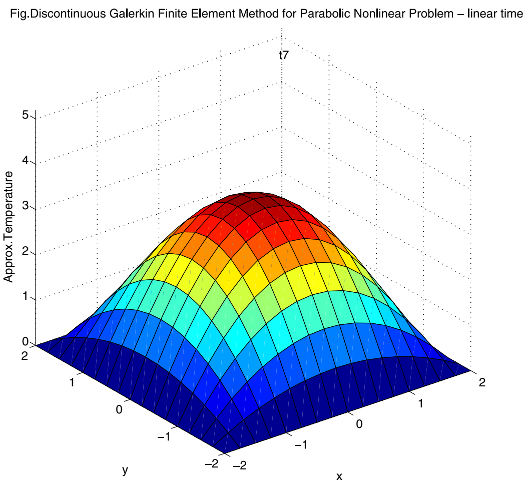
We assume boundary as well as initial conditions to be exact derived from u. There are a total of 256 finite elements taken over the region Ω and the tolerance $\delta = 0.005$ is used for convergence. The program also set the maximum number of iterations to be 5, for convenience, and computation moves forward to the next time level, when the maximum number of iterations are performed. This device is put into effect to prevent unnecessary iterations if approximations are no longer improving. The overall error is a combination of modelling error and discretization error. Moreover, with the parabolic problems, discretization error accumulates linearly as time moves forward. Thus, if iterations terminate before tolerance is achieved and one requires more accuracy in computation, then some refinements of the finite elements are required. We used the linear splines to approximate u in space and time variables.

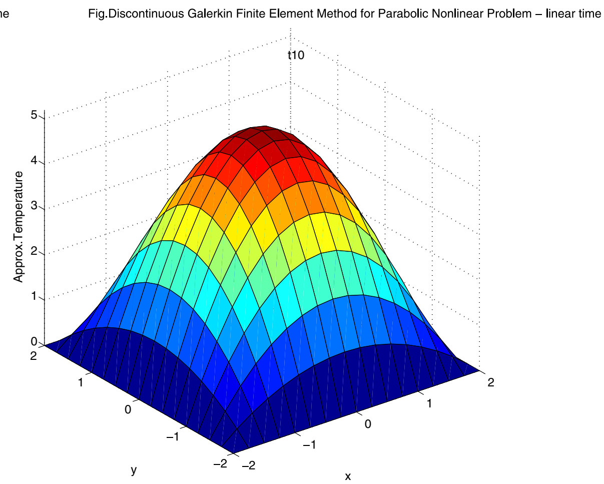
t	Example 1	
	Count	Error
t_1	1	0.0016
t_2	1	0.0029
t_3	1	0.0042
t_4	2	0.0049
t_5	5	0.0058
t_6	5	0.0066
t_7	5	0.0074
t_8	5	0.0083
t_9	5	0.0091
t_{10}	5	0.0010

The temperature profiles at four different time levels, t_1, t_3, t_7 and t_{10} are given below.









Please cite this article in press as: H. Kaneko et al., Numerical experiments using hierarchical finite element ..., Appl. Math. Comput. (2008), doi:10.1016/j.amc.2007.12.062

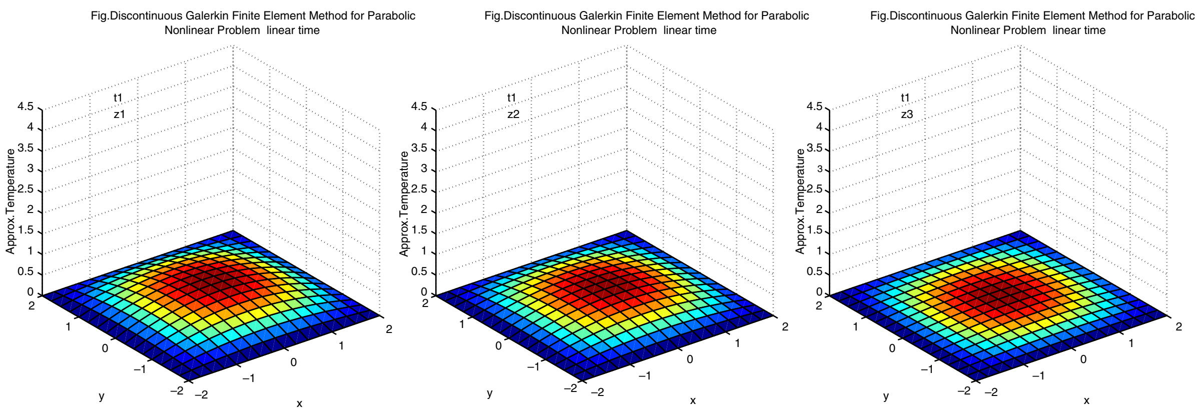
Example 2. Now we consider (2.1) over three-dimensional plate $\Omega = [-2,2] \times [-2,2] \times [-1,1]$. Assume the same number 256 elements over the surface $[-2,2] \times [-2,2]$ and a set of hierarchical functions $\{1, z^2, z^4\}$ through the thickness of plate $[-1,1]$. Linear splines are used for approximating u in the x, y and t variables. The solution is taken to be

$$u(x,y,z,t) = (1 - 0.25x^2)(1 - 0.25y^2)(1 - z^2)^2 t$$

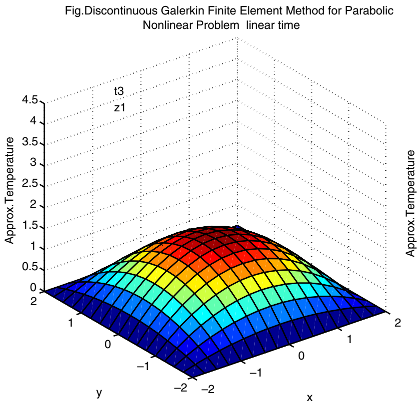
with $k(u) = 0.005u + 1$. Time interval of $[0,5]$ is taken and it is partitioned uniformly into 10 intervals. Errors in temperature at the cross sections of $z_1 = 0$, $z_2 = 1/3$ and $z_3 = 2/3$ are given below. Because of the symmetry of this problem, the temperatures at $z = -1/3$ and $z = 2/3$ follow from these data.

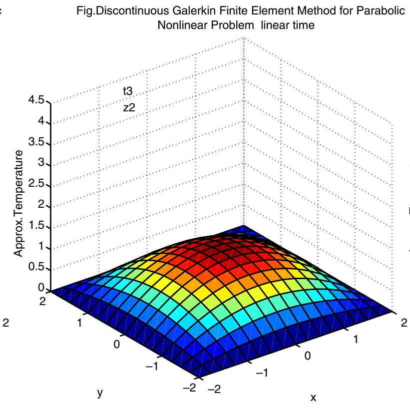
t	Example 2			
	Count	Error at z_1	Error at z_2	Error at z_3
t_1	1	0.0016	0.0012	0.0004
t_2	1	0.0032	0.0024	0.0006
t_3	1	0.0048	0.0036	0.0008
t_4	2	0.0056	0.0042	0.0011
t_5	3	0.0069	0.0051	0.0012
t_6	3	0.0081	0.0060	0.0013
t_7	3	0.0093	0.0069	0.0013
t_8	3	0.0106	0.0078	0.0014
t_9	3	0.0118	0.0087	0.0015
t_{10}	3	0.0130	0.0095	0.0016

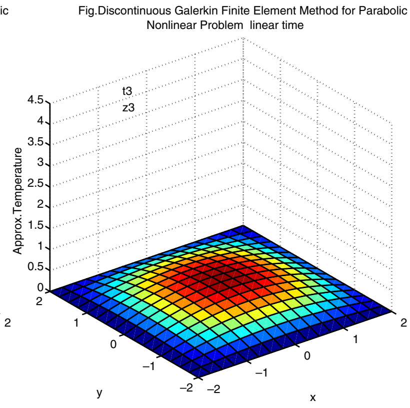
The temperature profiles over three cross sections $z_0 = 0, z_1 = 1/3$ and $z_2 = 2/3$ at four different time levels t_1, t_3, t_7 and t_{10} are drawn below.

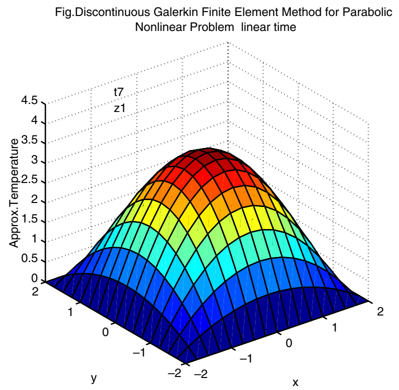


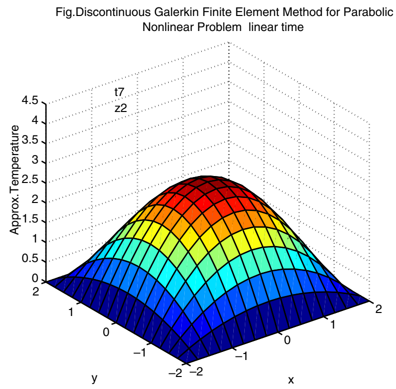
Please cite this article in press as: H. Kaneko et al., Numerical experiments using hierarchical finite element ..., Appl. Math. Comput. (2008), doi:10.1016/j.amc.2007.12.062

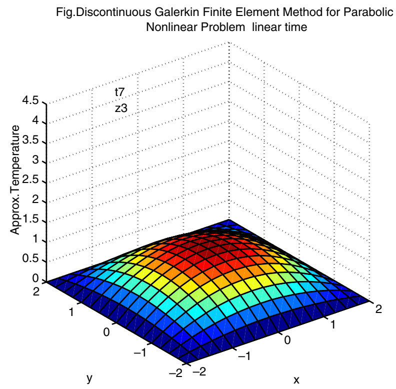


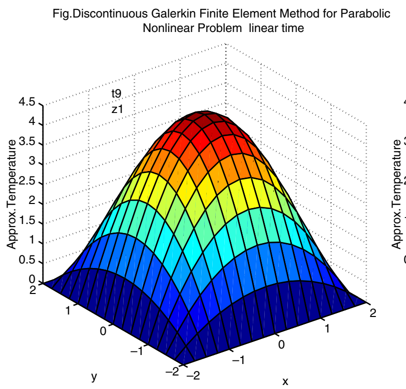


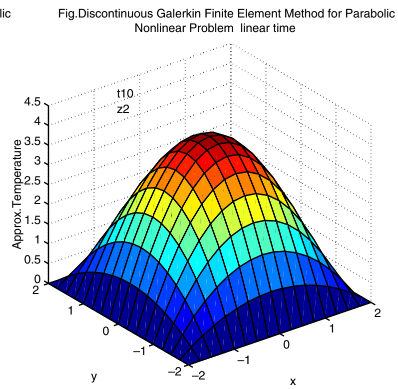


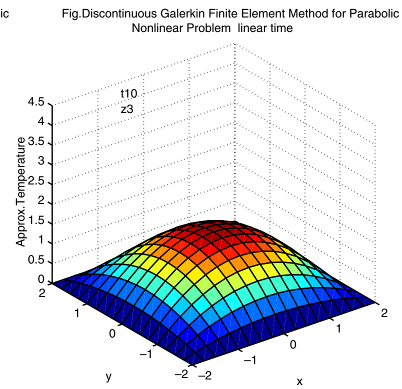












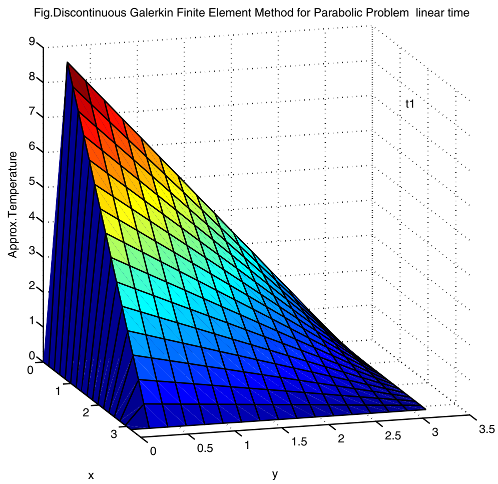
Example 3. The nonlinear parabolic problems (2.1) having boundary layers can be handled by the current method of ‘linearlization’. Suppose we consider (2.1) over two-dimensional plate $\Omega = [0, \pi] \times [0, \pi]$ with boundary conditions

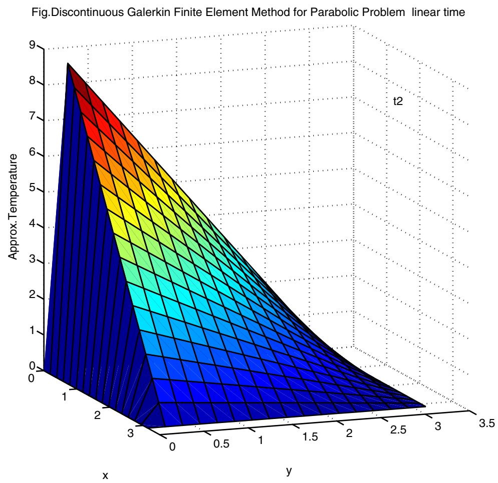
$$u = 0 \quad \text{on } \partial\Omega \tag{3.6}$$

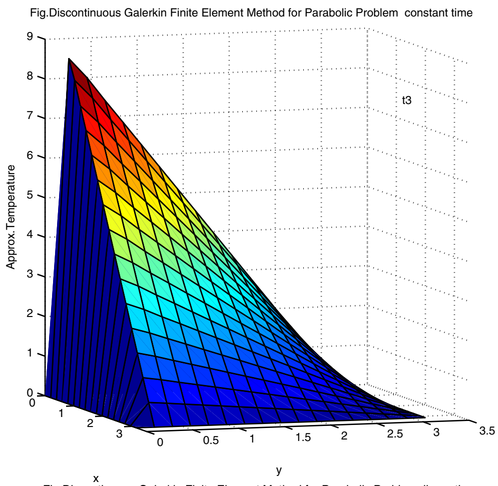
and initial condition

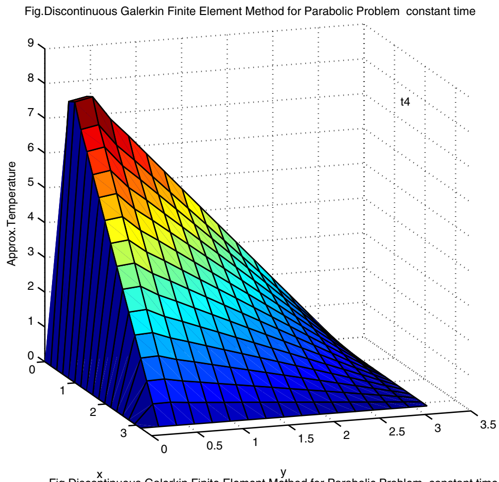
$$u(x, y, 0) = (\pi - x)(\pi - y). \tag{3.7}$$

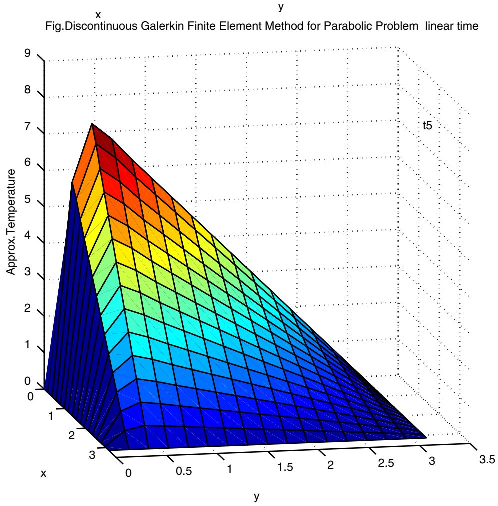
Then it is clear that we have a boundary region along $x = 0$ and $y = 0$ in which we have a solution changing rapidly neat $t = 0$, due to the incompatibility between the boundary condition and the initial condition, i.e., the transient phase of the solution. The current method works well for this type of nonlinear parabolic problems over a two-dimensional plate region. Here are the temperature profiles at eight different time levels (non-uniform) of (2.1) with boundary condition (3.6) and initial condition (3.7) imposed. Some mesh refinements within the region boundary layers are done to maintain accurate approximation.

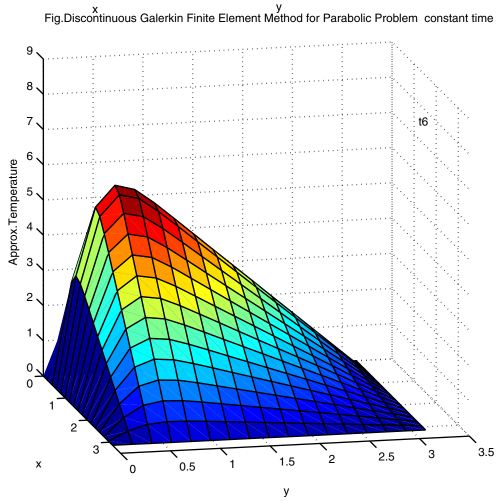


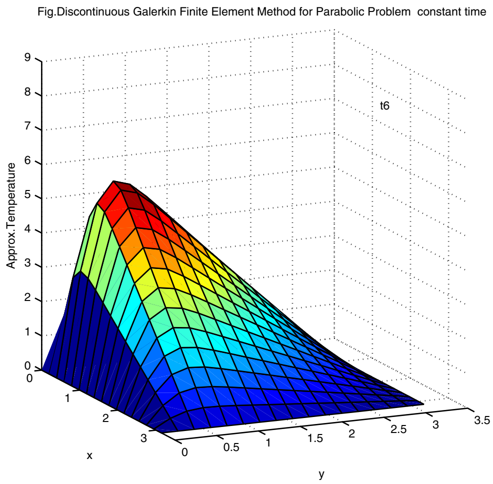


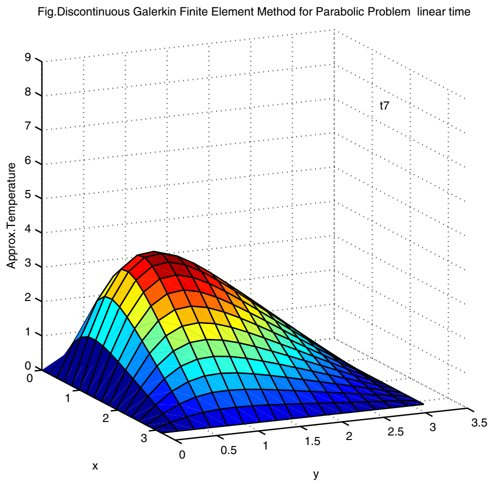












Example 4. Now we consider the heat conduction problem over a three-dimensional multilayered plate where each layer has different heat conductivities. In the case of a plate having constant conductivities over each layer, hierarchical basis functions are piecewise polynomials. Hence, in order to construct basis functions for hierarchical modelling for heat conduction problem through the thickness of a multilayered plate, it is ideal that these basis functions satisfy the temperature and the flux conditions across each boundary between two layers. Typically, for example, problem (2.1) over a multilayered plate is endowed with the following additional conditions:

$$u_i(\cdot, Z_i) = u_{i+1}(\cdot, Z_i), \quad i = 1, \dots, n_L, \tag{3.6}$$

$$k_{z,i}\frac{\partial u_i}{\partial z}(\cdot, Z_i) = k_{z,i+1}\frac{\partial u_{i+1}}{\partial z}(\cdot, Z_i), \quad i = 1, \dots, n_L - 1. \tag{3.7}$$

Hence, one would like to have hierarchical basis functions to satisfy these conditions. To this end, let

$$\varphi_0(\eta) = \frac{1}{2}(1-\eta),$$

$$\begin{aligned}\varphi_1(\eta) &= \frac{1}{2}(1+\eta),\\ \varphi_i(\eta) &= \sqrt{\frac{2i-1}{2}}\int_{-1}^{\eta} P_{i-1}(\eta)\mathrm{d}\xi,\\ &= \frac{1}{\sqrt{2(2i-1)}}(P_i(\eta) - P_{i-2}(\eta)), \quad i = 2, 3, \dots, p_x,\end{aligned} \tag{3.8}$$

where P_i is the Legendre polynomial of the first kind of order i given by

$$P_0(\eta) = 1,$$

$$P_1(\eta) = \eta,$$

$$P_i(\eta) = \frac{1}{i}[(2i-1)\eta P_{i-1}(\eta) - (i-1)P_{i-2}(\eta)], \quad i = 2, 3, \dots,$$

$$\psi_0(y) = 1,$$

$$\psi_j(y) = \begin{cases} \psi_j^1(y) & -1 < y < \eta_1, \\ \psi_j^2(y) & \eta_1 < y < \eta_2, \\ \vdots & \\ \psi_j^{n_L}(y) & \eta_{n_L-1} < y < 1, \end{cases} \qquad j = 1, 2, \ldots,$$

where

$$\psi_j^1(y) = \frac{1}{k_1}\varphi_j(y),$$
$$\psi_j^i(y) = \frac{1}{k_i}(\varphi_j(y) - \varphi_j(\eta_{i-1})) + c_i, \quad i = 2, \ldots, n_L$$

and

$$\begin{aligned} c_2 &= \frac{1}{k_1}\varphi_j(\eta_1), \\ c_{i+1} &= c_i + \frac{1}{k_i}(\varphi_j(\eta_i) - \varphi_j(\eta_{i-1})), \quad i = 2, \ldots, n_L - 1, \end{aligned} \tag{3.9}$$

ψ_j's are thus hierarchical and it is not difficult to see that they satisfy (3.6) and (3.7). Now, we utilize the construction method just described in (3.9) to (2.1) with the following characteristics. First, we assume that $\Omega = [-2, 2] \times [-2, 2] \times [-d, d]$, $[0, T] = [0, 5]$ and the solution is taken to be

$$u(x, y, z, t) = \begin{cases} (1 - 0.25x^2)(1 - 0.25y^2)\frac{(d^2-z^2)^2}{kz_1}t, & -d \leqslant z < 0, \\ (1 - 0.25x^2)(1 - 0.25y^2)\left(\frac{((d^2-z^2)^2-d^4)}{kz_2} + \frac{d^4}{kz_1}\right)t, & 0 \leqslant z < d, \end{cases}$$

where $kz_1 = kz_{1ij} = 0.005u(x_i, y_j, 0, t_{n-1}) + 1$ and $kz_2 = kz_{2ij} = 0.05u(x_i, y_j, 0, t_{n-1}) + 10$ at each node (x_i, y_j) and

$$k(u) = \begin{cases} 0.005u + 1, & -d \leqslant z < 0, \\ 0.05u + 10, & 0 \leqslant z < d. \end{cases}$$

The total of $N = 256$ elements are taken over $[-2, 2] \times [-2, 2]$ and a set of hierarchical basis functions $\{\psi_{0ij}(z), \psi_{1ij}(z), \psi_{2ij}(z)\}$ are defined according to (3.9). In particular, we define

$$\psi_{0ij}(z) = 1, \quad \psi_{1ij}(z) = \begin{cases} \frac{z^2}{kz_{1ij}}, & -d < z < 0, \\ \frac{z^2}{kz_{2ij}}, & 0 < z < d, \end{cases}$$

$$\psi_{2ij}(z) = \begin{cases} \frac{z^4}{kz_{1ij}}, & -d < z < 0, \\ \frac{z^4}{kz_{2ij}}, & 0 < z < d. \end{cases}$$

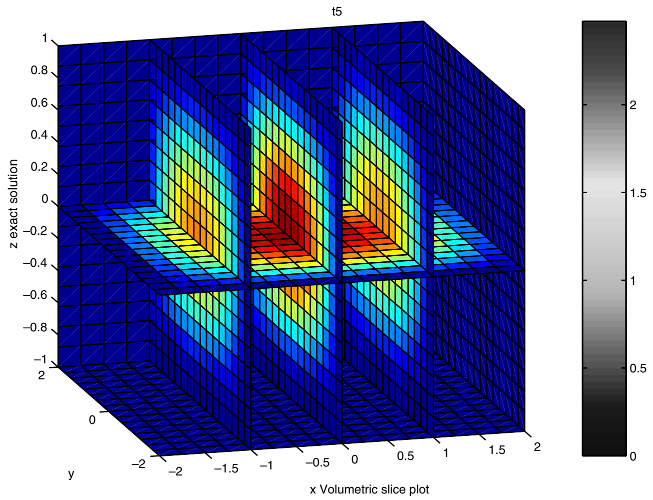
We take $d = 1$ in the next computations. Errors in the temperature at the cross sections of $z_1 = 0, z_2 = 1/3$ and $z_3 = 2/3$ are given below.

Errors in temperature through the thickness and at z_1, z_2 and z_3

t	Count	Error through the thickness	Error at z_1	Error at z_2	Error at z_3
0.5	1	0.00163	0.00146	0.00142	0.00133
1.0	1	0.00395	0.00355	0.00345	0.00321
1.5	3	0.00626	0.00562	0.00545	0.00507
2.0	3	0.00995	0.00891	0.00864	0.00803
2.5	3	0.01441	0.01288	0.01250	0.01162

Please cite this article in press as: H. Kaneko et al., Numerical experiments using hierarchical finite element ..., Appl. Math. Comput. (2008), doi:10.1016/j.amc.2007.12.062

The following is a volumetric slice plot of approximate temperature at time step $t5 = 2.5$ for this two-layered plate parabolic problem.



References

[1] H. Kaneko, K.S. Bey, G.J. Hou, discontinuous Galerkin finite element method for parabolic problems, Appl. Math. Comput. 182 (2006) 388–402.

[2] H. Kaneko, K.S. Bey, G.J. Hou, A discontinuous Galerkin method for parabolic problems with modified hp finite element approximation technique, Appl. Math. Comput. 182 (2006) 1405–1417.

[3] H. Kaneko, G.J. Hou, Discontinuous Galerkin finite element method for nonlinear parabolic problems with non-smooth data, an invited paper in Far East J. Appl. Math. 14 (2004) 137–155.

[4] K. Eriksson, C. Johnson, Error estimates and automatic time step control for nonlinear parabolic problems, I, SIAM J. Numer. Anal. 24 (1) (1987) 12–23.

[5] K. Eriksson, C. Johnson, Adaptive finite element methods for parabolic problems I: a linear model problem, SIAM J. Numer. Anal. 28 (1) (1991) 43–77.

[6] K. Eriksson, C. Johnson, Adaptive finite element methods for parabolic problems IV: nonlinear problems, SIAM J. Numer. Anal. 32 (6) (1995) 1729–1749.

[7] K. Eriksson, C. Johnson, A. Logg, Adaptive computational methods for parabolic problems, in: E. Stein, R. de Borst, T.J.R. Hughes (Eds.), Encyclopedia of Computational Mechanics, John Wiley & Sons, Ltd., 2004, pp. 1–44.

[8] S. Jensen, I. Babuska, Dimension reduction for nonlinear boundary value problems, SIAM J. Numer. Anal. 25 (3) (1988) 644–669.

[9] S. Jensen, Adaptive dimensional reduction numerical solution of monotone quasilinear boundary value problems, SIAM J. Numer. Anal. 29 (5) (1992) 1294–1320.

[10] C. Johnson, Numerical Solution of Partial Differential Equations by the Finite Element Methods, Cambridge University Press, 1994.

[11] C. Schwab, Boundary layer resolution in hierarchical models of laminated composites, Math. Modell. Numer. Anal. 28 (1994) 517–537.

[12] C. Schwab, Boundary layer resolution in hierarchical plate modelling, Math. Methods Appl. Sci. 18 (1995) 345–370.

[13] D. Schotzau, C. Schwab, Time discretization of parabolic problems by the hp-version of the discontinuous Galerkin finite element methods, SIAM J. Numer. Anal. 38 (2000) 837–875.

[14] D. Schotzau, C. Schwab, An hp a priori error analysis of the DG time stepping methods for initial value problems, Calcolo 37 (2000) 207–232.

[15] M. Vogelius, I. Babuska, On a dimensional reduction model, I. The optimal selection of basis functions, Math. Comput. 37 (1981) 31–46.

[16] M. Vogelius, I. Babuska, On a dimensional reduction model, II. Some approximation – theoretical results, Math. Comput. 37 (1981) 47–68.