ON A VARIATIONAL PRINCIPLE OF EKELAND

Peter Z. Daffer
Department of Mathematics
Macon College
Macon, Georgia 31297 USA

Hideaki Kaneko
Department of Mathematics and Statistics
Old Dominion University
Norfolk, Virginia 23529-0077 USA

Abstract

In a well known paper [3], Ekeland presented a variational principle that can be used for many useful applications. In a recent paper [2], the present authors showed that the widely recognized fixed point theorem of Caristi [1] and the minimization principle of Takahashi [5] are indeed equivalent, thereby proving the equivalence between these theorems by Ekeland, Caristi and Takahashi. In this paper, we further investigate the relationship among these theorems by characterizing the set of minimizers, the set of variational points and the set of fixed points.

AMS subject classifications (1991). Primary 47H10.
Key words and phrases. Fixed point theorems, Variational principle, Minimization principle, Set of minimizers.

1 Introduction

In a well known paper [3], Ekeland presented a variational principle which contains numerous useful applications. The main result of a fundamental nature that Ekeland presented is the following:

Theorem 1.1 *Let (X, d) be a complete metric space and $\varphi: X \to (-\infty, \infty]$ a proper lower semicontinuous function bounded from below (here 'proper' means φ is not identically equal to*

∞). Let $\epsilon > 0$ be given and a point $u \in X$ such that

$$\varphi(u) \leq \inf_{x \in X} \varphi(x) + \epsilon.$$

Then there exists some point $v \in X$ such that

$$\begin{aligned} \varphi(v) &\leq \varphi(u) \\ d(u,v) &\leq 1 \\ \varphi(w) &> \varphi(v) - \epsilon d(v,w) \qquad \textit{for all } w \neq v. \end{aligned}$$

In [3], a wide range of applications of Theorem 1.1 are given. One application that Ekeland mentions is the fixed point of Caristi [1]. In 1976, Caristi published a paper in which he presented a fixed point theorem that requires no continuity of the mapping under consideration. Caristi's theorem is the following:

Theorem 1.2 *Let (X, d) be a complete metric space and $\varphi: X \to (-\infty, \infty]$ a proper lower semicontinuous function bounded from below. Let $f: X \to X$ satisfy*

$$d(x, f(x)) \leq \varphi(x) - \varphi(f(x)) \qquad \textit{for all } x \in X.$$

Then f has a fixed point in X, -i.e. there exists $x \in X$ such that $f(x) = x$.

It has been observed that Theorems 1.1 and 1.2 are equivalent, -e.g. see [4]. In the other area of application of Theorem 1.1, Takahashi [5] proved an existence theorem for a certain class of nonconvex minimization problems. What Takahashi proved in [5] is the following:

Theorem 1.3 *Let (X, d) be a complete metric space and let $\varphi: X \to (-\infty, \infty]$ be a proper lower semicontinuous function, bounded from below. Suppose that, for each $u \in X$ with $\varphi(u) > \inf_{x \in X} \varphi(x)$, there is a $v \in X$ such that $v \neq u$ and $\varphi(v) + d(u, v) \leq \varphi(u)$. Then there exists an $x_0 \in X$ such that $\varphi(x_0) = \inf_{x \in X} \varphi(x)$.*

Takahashi then observed that Theorem 1.3 contains Theorems 1.1 and 1.2 as its corollaries. Recently, the present authors proved in [2] that Theorem 1.2 of Caristi and Theorem 1.3 of Takahashi are indeed equivalent, thereby demonstrating that all three theorems above are equivalent. The purpose of this paper is to advance the study which began in [2]. Specifically, we investigate the relationship among Theorems 1.1-1.3 by examining the interplay between the minimizer of φ, M_φ and the set Φ_ϵ (see below for its definition). This will be done in Section 2. In Section 3, we examine the relationship between these sets M_φ, Φ_ϵ and the set $F(f)$ of fixed points for a mapping f that satisfies the conditions of Caristi's theorem (Theorem 1.2).

2 Variational Points and Minimizers

Let (X, d) be a complete metric space, and $\varphi\colon X \to (-\infty, +\infty]$ a proper lower semi-continuous mapping bounded from below. Define the minimum set of φ as

$$M_\varphi = \{x \in X\colon \varphi(x) = \inf_{y \in X} \varphi(y)\}. \tag{2.1}$$

Also define the set

$$\Phi_\epsilon = \{x \in X\colon \varphi(y) + \epsilon d(x, y) > \varphi(x), \text{ for all } y \neq x\}. \tag{2.2}$$

By Theorem 1.1, we see that

$$\Phi_\epsilon \neq \emptyset \qquad \text{for every } \epsilon > 0. \tag{2.3}$$

Moreover the following are clear,

$$\Phi_{\epsilon'} \subset \Phi_\epsilon \qquad \text{whenever } 0 \leq \epsilon' \leq \epsilon \tag{2.4}$$

$$M_\varphi \subset \Phi_\epsilon \qquad \text{for every } \epsilon > 0. \tag{2.5}$$

Proposition 2.1 *If M_φ contains two or more points, then $\Phi_0 = \emptyset$. Consequently, in general $\Phi_0 \neq \cap_{\epsilon>0}\Phi_\epsilon$.*

Proof: Suppose there are two distinct points $x, y \in M_\varphi$ and that $z \in \Phi_0$. Then $\varphi(u) > \varphi(z)$ for all $u \neq z$. But this fails for u equal to at least one of the points x or y. Moreover, if x, $y \in M_\varphi$, then $x, y \in \Phi_\epsilon$ for every $\epsilon > 0$ while $\Phi_0 = \emptyset$. Of course, we always have $\Phi_0 \subset \cap_{\epsilon>0}\Phi_\epsilon$. $\square$.

Proposition 2.2 $\Phi_0 \subset M_\varphi \subset \Phi_\epsilon$ *for every* $\epsilon > 0$.

Proof: The second set inclusion was observed in (2.5). As to the first set inclusion, if $\Phi_0 = \emptyset$, then the result is trivial. If $x \in \Phi_0$, then $\varphi(y) > \varphi(x)$, for all $y \neq x$ and so $\varphi(x) = \inf_{y \in X} \varphi(y)$ and $x \in M_\varphi$. $\square$

Having made preliminary observations concerning the relationship between M_φ and Φ_ϵ, we are in a position to present the following theorem which generalizes Theorem 1.3 of Takahashi.

Theorem 2.3 *Let (X, d) be a complete metric space and let $\varphi\colon X \to (-\infty, \infty]$ be a proper lower semicontinuous function, bounded from below. Suppose that, for each $u \in X$ with $\varphi(u) > \inf_{x \in X} \varphi(x)$, there is a $v \in X$ such that $v \neq u$ and $\varphi(v) + \epsilon d(u, v) \leq \varphi(u)$ with $\epsilon > 0$. Then there exists an $x_0 \in X$ such that $\varphi(x_0) = \inf_{x \in X} \varphi(x)$.*

A constructive proof demonstrated in [2] for Theorem 1.3 can be modified here. Also, one can simply re-metrize the space X with ϵd and obtain a proof from Takahashi's original formulation [5]. We first note that using the sets defined thus far, the hypothesis of Theorem 1.3 states that

$$x \notin M_\varphi \Longrightarrow x \notin \Phi_1. \tag{2.6}$$

which is equivalent to $\Phi_1 \subset M_\varphi$ and Theorem 1.3 of Takahashi can be reformulated as

$$\Phi_1 \subset M_\varphi \Longrightarrow M_\varphi \neq \emptyset. \tag{2.7}$$

Similarly, Theorem 2.3 can be characterized as

$$\Phi_\epsilon \subset M_\varphi \text{ for some } \epsilon > 0 \Longrightarrow M_\varphi \neq \emptyset. \tag{2.8}$$

Set-theoretically, (2.8) is also clear, since by Theorem 1.1 of Ekeland, we have $\Phi_\epsilon \neq \emptyset$ for all $\epsilon > 0$ and thus $M_\varphi \neq \emptyset$ under the hypothesis of Theorem 2.3. In fact, using Proposition 2.2, we obtain, if, for some $\epsilon > 0$, $\Phi_\epsilon \subset M_\varphi$, then

$$\Phi_\epsilon = M_\varphi. \tag{2.9}$$

One may enhance the characterization of Theorem 2.3 that was made in (2.8). In order to do this, we make the following definition:

$$\epsilon_0 = \sup\{\epsilon \geq 0 \colon \Phi_\epsilon \subset M_\varphi\}. \tag{2.10}$$

Since $\Phi_0 \subset M_\varphi$, ϵ_0 is well defined and $\epsilon_0 \in [0, +\infty]$. Using ϵ_0, Theorem 2.3 can be reformulated as:

$$\epsilon_0 > 0 \Longrightarrow M_\varphi \neq \emptyset. \tag{2.11}$$

Lemma 2.4 *If $\epsilon_0 > 0$, then $\Phi_\epsilon = M_\varphi$ for every $\epsilon \in (0, \epsilon_0)$.*

Proof: By (2.4) and (2.11), if $\epsilon \in (0, \epsilon_0)$, then $\Phi_\epsilon \subset M_\varphi$. Hence the result follows by noting Proposition 2.2. $\square$

Having made some observations concerning the sets Φ_ϵ and M_φ, the equivalence of Theorem 1.1 of Ekeland and Theorem 2.3 can be seen.

Theorem 2.5 *Theorem 1.1 and Theorem 2.3 are equivalent.*

Proof: By Theorem 1.1 of Ekeland, $\Phi_\epsilon \neq \emptyset$ for $\epsilon > 0$. Hence

$$M_\varphi = \emptyset \Longrightarrow \epsilon_0 = 0. \tag{2.12}$$

But the contrapositive of (2.12) is precisely Theorem 2.3 as in (2.11). The converse is similar. □

With Theorem 2.3, a generalization of Takahashi's Theorem 1.3, formulated as (2.11), one could ask whether the converse holds. The converse would read

$$M_\varphi \neq \emptyset \Longrightarrow \epsilon_0 > 0. \tag{2.13}$$

This implication turns out to be false and the following counterexample demonstrates it.

Counterexample: In this example we present a situation where $M_\varphi \neq \emptyset$ but for every $\epsilon > 0$, Φ_ϵ contains points which are not in M_φ. Let $X = [1, \infty) \cup \{-1\}$ be endowed with the usual metric on R. Define

$$\varphi(x) = \begin{cases} \frac{1}{x}, & \text{for } x \geq 1 \\ 0, & \text{for } x = -1 \end{cases}$$

We have $M_\varphi = \{-1\}$ and it is not difficult to show that for each $\epsilon > 0$, $\epsilon^{-1/2} \in \Phi_\epsilon$.

3 Caristi's Fixed Point Theorem

In this section, we invesigate the relationship between the sets Φ_ϵ, M_φ and the set $F(f)$ of fixed points of f that satisfies the conditions of Theorem 1.2 of Caristi. As before, we let (X, d) be a complete metric space and $\varphi\colon X \to (-\infty, \infty]$ a proper lower semicontinuous function bounded from below. Let $f\colon X \to X$ satisfy

$$d(x, f(x)) \leq \varphi(x) - \varphi(f(x)) \qquad \text{for all } x \in X. \tag{3.1}$$

Define C_φ to be the set of all $f\colon X \to X$ for which the condition (3.1) holds for φ. Then Theorem 1.2 can be formulated as

$$f \in C_\varphi \Longrightarrow F(f) \neq \emptyset. \tag{3.2}$$

First we obtain,

Lemma 3.1 *If $f \in C_\varphi$, then $\Phi_\epsilon \subset F(f)$ for $0 \leq \epsilon \leq 1$.*

Proof: For each arbitrary but fixed $\epsilon \in [0, 1]$, by Theorem 1.1, $\Phi_\epsilon \neq \emptyset$. Let $x \in \Phi_\epsilon$. If $x \neq f(x)$, then we get $\varphi(x) - \varphi(f(x)) < \epsilon d(x, f(x))$, while, on the other hand, $d(x, f(x)) \leq \varphi(x) - \varphi(f(x))$. For $\epsilon \leq 1$, this is a contradiction. Hence $x \in F(f)$. □

Summarizing what we have so far

$$f \in C_\varphi \Longrightarrow \Phi_0 \subset M_\varphi \subset \Phi_\epsilon \subset F(f), \qquad \text{for } \epsilon \in [0, 1]. \tag{3.3}$$

In particular, the set M_φ of minimizers of φ consists of fixed points of f. If the hypothesis of Theorem 2.3 holds, then we have

$$\epsilon_0 > 0 \quad \text{and} \quad f \in C_\varphi \Longrightarrow \Phi_0 \subset M_\varphi = \Phi_\epsilon \subset F(f), \tag{3.4}$$

for $0 < \epsilon < \min\{1, \epsilon_0\}$. Some further observations of interest are that an example shows that one can have

$$\Phi_0 = M_\varphi = \Phi_\epsilon \subset F(f), \quad \text{with } \Phi_\epsilon \neq F(f), \text{ for every } 0 \leq \epsilon < \infty$$

(here $\epsilon_0 = +\infty$) and that another example shows that

$$\{0\} = \Phi_0 = M_\varphi \subset \Phi_\epsilon \subset F(f), \quad \text{for } \epsilon > 0 \text{ and } f \in C_\varphi.$$

Lemma 3.2 *If $f \in C_\varphi$, then for $0 \leq \epsilon < \epsilon_0$ we have $\Phi_\epsilon \subset F(f)$.*

Proof: This is clear since $\Phi_\epsilon \subset M_\varphi \subset F(f)$ for $0 \in [0, \epsilon_0)$. $\square$

We observe that if $\epsilon_0 \leq 1$, then lemma 3.1 is better that lemma 3.2 whereas for $\epsilon_0 > 1$, lemma 3.2 is better. Now we expand on the Caristi's condition. Namely consider the following condition; for $f\colon X \to X$ and $0 \leq \epsilon < \infty$,

$$\varphi(f(x)) + \epsilon d(x, f(x)) \leq \varphi(x). \tag{3.5}$$

We write C_ϵ^φ for the set of all f which satisfy the condition (3.5). $C_\epsilon^\varphi \neq \emptyset$ since it contains the identity function.

Proposition 3.3 *If $f \in C_\epsilon^\varphi$ for some $\epsilon \in (0, \infty)$, then $F(f) \neq \emptyset$.*

Proof: Simply renorm the metric space with the equivalent norm ϵd and apply Theorem 1.2 of Caristi to $(X, \epsilon d)$. $\square$

Now we define

$$\begin{aligned} \epsilon_f &= \sup\{\epsilon \geq 0\colon \varphi(f(x)) + \epsilon d(x, f(x)) \leq \varphi(x), \quad \text{for all } x \in X\} \\ &= \sup\{\epsilon \geq 0\colon f \in C_\epsilon^\varphi\}. \end{aligned}$$

We first note that C_ϵ^φ is antitone, -i.e. $C_{\epsilon'}^\varphi \subset C_\epsilon^\varphi$ for $0 < \epsilon \leq \epsilon'$. Therefore $f \in C_\epsilon^\varphi$ for every ϵ with $0 < \epsilon \leq \epsilon_f$.

Proposition 3.4 *If $f \in C_\epsilon^\varphi$ for some $\epsilon > 0$, then $\Phi_\epsilon \subset F(f)$ for $0 \leq \epsilon \leq \epsilon_f$.*

Proof: For every $\epsilon > 0$, by Theorem 1.1 of Ekeland $\Phi_\epsilon \neq \emptyset$. If $x \in \Phi_\epsilon$ and $x \neq f(x)$, then we get $\varphi(x) - \varphi(f(x)) < \epsilon d(x, f(x))$. By hypothesis, $\epsilon_f > 0$ and for $0 < \epsilon \leq \epsilon_f$, we have $\epsilon d(x, f(x)) \leq \varphi(x) - \varphi(f(x))$, and in particular, $\epsilon_f d(x, f(x)) \leq \varphi(x) - \varphi(f(x))$ for all $x \in X$. This contradicts the first inequality if $\epsilon \leq \epsilon_f$, and hence in this case, we have $x = f(x)$, i.e., $x \in F(f)$. $\square$

REMARK An example shows that there is φ such that, for $\epsilon > 0$, C_ϵ^φ is a singleton set, consisting of the identity function alone. In this case, we have $\epsilon_f = 0$. If, for a function f, we have $f \notin C_\epsilon^\varphi$ for every $0 \leq \epsilon < \infty$, then we take $\epsilon_f = 0$. Note that $C_\infty^\varphi = \emptyset$ always, since φ is assumed to be proper.

In fact, Proposition 3.4 can be improved. Since we always have $M_\varphi \subset F(f)$, if $\epsilon_0 > 0$ (recall (2.10)), we have $0 < \epsilon < \epsilon_0 \Longrightarrow \Phi_\epsilon \subset F(f)$ by Lemma 2.4. If $\epsilon_f > 0$ (so that $f \in C_{\epsilon_f}^\varphi$), we have $0 < \epsilon \leq \epsilon_f \Longrightarrow \Phi_\epsilon \subset F(f)$. Together, these observations yield:

Proposition 3.5 *If* $\max\{\epsilon_0, \epsilon_f\} > 0$*, then for* $0 < \epsilon < \max\{\epsilon_0, \epsilon_f\}$ *we obtain* $\Phi_\epsilon \subset F(f)$*.*

Lemma 3.6 *For a given* φ*, and any* f *for which* $\epsilon_f > 0$ *(so that* $f \in C_{\epsilon_f}^\varphi$*), we have* $\epsilon_0 \leq \epsilon_f$*.*

Proof: Suppose the contrary that $0 < \epsilon_f < \epsilon_0$. Then for $\epsilon_f < \epsilon < \epsilon_0$, choose $x \in \Phi_\epsilon$. Since $\epsilon < \epsilon_0$, we have $x \in \Phi_\epsilon \subset M_\varphi \subset F(f)$, so that $f(x) = x$. But since $\epsilon > \epsilon_f$, there is some $x \in X$ such that $\varphi(f(x)) + \epsilon d(x, f(x)) > \varphi(x)$. This contradicts $f(x) = x$. $\square$

Thus, for any lower semicontinuous proper function φ that is bounded below, the number ϵ_0 provides a lower bounded for ϵ_f for any self-map f which satisfies the hypothesis of Caristi's fixed point theorem. Proposition 3.5 can now be improved as follows:

Proposition 3.7 *If* $\epsilon_f > 0$*, then* $\Phi_\epsilon \subset F(f)$ *for all* $0 \leq \epsilon \leq \epsilon_f$*.*

References

[1] J. Caristi, *Fixed Point Theorems for Mappings Satisfying Inwardness Conditions,* Trans. Amer. Math. Soc. 215 (1976), 241-251.

[2] P. Z. Daffer and H. Kaneko, *On the Variational Principle,* submitted to Jl. Math. Anal. and Appl.

[3] I. Ekeland, *Nonconvex Minimization Problems,* Bull. Amer. Math. Soc. (New Series) 1 (1979), 443-474.

[4] N. Mizoguchi and W. Takahashi, *Fixed Point Theorems for Multivalued Mappings on Complete Metric Spaces,* Jl. Math. Anal. Appl. 141 (1989), 177-188.

[5] W. Takahashi, *Existence Theorems Generalizing Fixed Point Theorems for Multivalued Mappings,* Fixed Point Theory and Applications, ed. M. A. Théra and J. B. Baillon, Longman Scientific and Technical, (1989) 397-406.