Taylor-series expansion methods for nonlinear Hammerstein equations

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Abstract

In this paper, we establish a simple yet effective Taylor series expansion method of approximating the solutions of nonlinear Hammerstein equations. The method lends itself to numerical computations which can be done in parallel. Numerical examples are provided to demonstrate the effectiveness of the current method.

1 Inroduction

In a recent paper [1], a Taylor-series expansion method to approximate the solution of a class of Fredholm integral equation of the second kind was considered. Subsequently, the present authors [2] generalized the result to obtain a new Taylor-series method which not only produces more accurate approximations but also can be applied to a wider class of Fredholm equations. Moreover, a numerical implementation of the new method can be carried out in parallel. This is an important point since most of the numerical methods available for approximating the solution of linear as well as nonlinear equations involve large scale linear or nonlinear systems which are mostly dense. Therefore, reducing an expensive cost involved in the solution process is always an important issue in approximating the solution of integral equations. The purpose of this paper is to extend the idea explored in [2] to nonlinear Hammerstein equations. A Hammerstein equation arises naturally as an integral equation reformulation of two-point boundary value problem with nonlinear boundary terms and it can be written as

$$x(s) - \int_0^1 k(s,t)\psi(t,x(t))dt = y(s), \qquad 0 \le s \le 1, \tag{1.1}$$

where ψ indicates a nonlinear term and x is to be determined. Also, throughout the paper, we assume that $k \in C^n([0,1]\times[0,1])$, $y \in C^n[0,1]$ and the following condition to hold,

(C) $\psi(t,x)$ is continuous in $t \in [0,1]$ and Lipschitz continuous in $x \in R$, and $\frac{\partial \psi}{\partial x}(t,x)$ exists and uniformly bounded over $[0,1]\times R$.

There have been a number of papers published recently to approximate the solution of equation (1.1). The reader may consult papers such as [3], [4], [5] and references cited therein for other different numerical techniques for approximating the solution of (1.1).

This paper is organized as follows. In Section 2, a new Taylor series method is applied directly to nonlinear Hammerstein equation. A resulting system of nonlinear equations is solved by the Newton's method. The Taylor series method lends naturally itself to parallel computation environment. In Section 3, we present a method which first transforms equation (1.1) into another nonlinear equation. Due to its special structure of this new equation, one can compute *a priori* key integrals only once throughout the entire solution process which reduces the computational cost. Computations in parallel are also possible in this context. Numerical examples are included at the end of each section.

2 Taylor-series Method for Hammerstein Equation:

In this section, we apply the Taylor expansion method developed in [2] to equation (1.1). We first write

$$x(t) \approx x(s) + x'(s)(t-s) + \cdots + \frac{1}{n!}x^{(n)}(s)(t-s)^n. \tag{2.1}$$

Substituting (2.1) for $x(t)$ in the integral in (1.1), we obtain

$$\begin{aligned} x(s) - \int_0^1 \quad & k(s,t)\psi(t, x(s) + x'(s)(t-s) + \cdots \\ & \cdots + \tfrac{x^{(n)}(s)}{n!}(t-s)^n)dt \approx y(s), \quad 0 < s < 1. \end{aligned} \tag{2.2}$$

This represents an nth order nonlinear differential equation with variable coefficients. It is possible that approximations to $x(s), x'(s), \ldots, x^{(n)}(s)$ can be found by solving (2.2) directly if suitable initial conditions are known for $x(0), x'(0), \ldots, x^{(n)}(0)$. The initial values may possibly be found from some experimental data, but in many instances, this is not the case. In the absence of this data, we proceed the solution process as follows. First, differentiate (1.1) n times, one obtains

$$\begin{aligned} x'(s) - \int_0^1 k_s'(s,t)\psi(t, x(t))dt \quad &= y'(s) \\ &\vdots \\ x^{(n)}(s) - \int_0^1 k_s^{(n)}(s,t)\psi(t, x(t))dt \quad &= y^{(n)}(s), \end{aligned} \tag{2.3}$$

where $k_s^{(i)}(s,t) = \partial^{(i)}k(s,t)/\partial s^i$, $i = 1, \ldots, n$. Next, each $x(t)$ in equations (2.3) is replaced by (2.1) to obtain, for $0 < s < 1$,

$$\begin{aligned} x'(s) - \int_0^1 k_s'(s,t)\psi(t, x(s) + x'(s)(t-s) + \tfrac{x^{(n)}(s)}{n!}(t-s)^n)dt \quad &\approx y'(s), \\ &\vdots \\ x^{(n)}(s) - \int_0^1 k_s^{(n)}(s,t)\psi(t, x(s) + x'(s)(t-s) + \tfrac{x^{(n)}(s)}{n!}(t-s)^n)dt \quad &\approx y^{(n)}(s). \end{aligned} \tag{2.4}$$

For simple k and ψ, equations (2.1) and (2.4) may be solved analytically for $x(s), x'(s), \ldots, x^{(n)}(s)$, but it is more likely that they must be solved numerically. To this end, let

$$\begin{aligned} f_i(x(s), x'(s), \ldots, \quad & x^{(n)}(s)) \equiv y^{(i)}(s) - x^{(i)}(s) + \int_0^1 k_s^{(i)}(s,t) \times \\ & \psi(t, x(s) + x'(s)(t-s) + \cdots + \tfrac{x^{(n)}(s)}{n!}(t-s)^n dt) \end{aligned}$$

for $i = 0, 1, \ldots, n$.

With

$$F(x(s), x'(s), \ldots, x^{(n)}(s)) \equiv \begin{bmatrix} f_0(x(s), x'(s), \ldots, x^{(n)}(s)) \\ f_1(x(s), x'(s), \ldots, x^{(n)}(s)) \\ \vdots \\ f_n(x(s), x'(s), \ldots, x^{(n)}(s)) \end{bmatrix}$$

$F(x(s), x'(s), \ldots, x^{(n)}(s)) = 0$ can be solved by the standard Newton method or by any other nonlinear equations solver. Note that in solving this system of nonlinear equations, approximations to $x(s), x'(s), \ldots, x^{(n)}(s)$ can be found for each s and since approximations to $x(s), x'(s), \ldots, x^{(n)}(s)$ for one value of s do not affect approximations to $x(s), x'(s), \ldots, x^{(n)}(s)$ for another s, the solution process can be done in parallel.

For the error associated with the proposed Taylor series expansion method, first we write equations in (1.1) and (2.3) as

$$x^{(i)}(s) - \int_0^1 k_s^{(i)}(s,t)\psi(t, \sum_{j=0}^{\infty} \frac{x^{(j)}(s)}{j!}(t-s)^j)dt = y^{(i)}(s), \quad 0 \le s \le 1 \quad (2.5)$$

for $i = 0, 1, \ldots, n$. Similarly $\bar{x}(s), \bar{x}'(s), \ldots, \bar{x}^{(n)}(s)$ denote the solution of equations (2.2) and (2.4), namely,

$$\bar{x}^{(i)}(s) - \int_0^1 k_s^{(i)}(s,t)\psi(t, \sum_{j=0}^{n} \frac{\bar{x}^{(j)}(s)}{j!}(t-s)^j)dt = y^{(i)}(s), \quad 0 \le s \le 1 \quad (2.6)$$

for $i = 0, 1, \ldots, n$.

From (2.5) and (2.6), and using condition **(C)** and the mean value theorem, we obtain

$$\begin{aligned} x^{(i)}(s) - \bar{x}^{(i)}(s) - &\int_0^1 k_s^{(i)}(s,t)\tfrac{\partial\psi}{\partial x}(t,\theta_i)\textstyle\sum_{j=0}^{n}(x^{(j)}(s) - \bar{x}^{(j)}(s))\tfrac{(t-s)^j}{j!}dt \\ &= \int_0^1 k_s^{(i)}(s,t)\tfrac{\partial\psi}{\partial x}(t,\theta_i)\tfrac{x^{(n+1)}(\xi_i(t))}{(n+1)!}(t-s)^{n+1}dt \end{aligned} \quad (2.7)$$

for some θ_i, $\xi_i(t)$, $0 \le s, t \le 1$ and $i = 0, 1, \ldots, n$. Let

$$\varepsilon_i(s) \equiv x^{(i)}(s) - \bar{x}^{(i)}(s)$$

$$a_{ij} \equiv \delta_{ij} - \int_0^1 k_s^{(i)}(s,t)\frac{\partial\psi}{\partial x}(t,\theta_i)\frac{(t-s)^j}{j!}dt \quad (2.8)$$

and

$$f_i \equiv \int_0^1 k_s^{(i)}(s,t)\frac{\partial\psi}{\partial x}(t,\theta_i)\frac{x^{(n+1)}(\xi_i(t))}{(n+1)!}(t-s)^{n+1}dt$$

for $i, j = 0, 1, \ldots, n$. Then, for each $0 \le s \le 1$, the error $\varepsilon_i(s)$ of Taylor series expansion method must satisfy the matrix equation

$$A_n\bar{\varepsilon}_n = F_n$$

where $A_n = [a_{ij}]$, $\bar{\varepsilon}_n = [\varepsilon_i]$ and $F_n = [f_i]$ for $i = 0, 1, \ldots, n$. Let $\|\cdot\|$ denote a vector norm as well as its corresponding matrix norm, then

$$\|\bar{\varepsilon}_n\| \leq \|A_n^{-1}\| \|F_n\|. \tag{2.9}$$

Example 2.1: We consider

$$x(s) - \frac{2}{\pi} \int_0^1 \frac{1}{4 + (s-t)^2} \sin x(t) dt = y(s)$$

and assume that y is selected so that $x(s) = 1 + s^2 + s^3$ is the solution. We note that a linear version of this equation, namely,

$$x(s) - \frac{a}{\pi} \int_0^1 \frac{1}{a^2 + (s-t)^2} x(t) dt = 1$$

is the classical Love's equation which plays an important role in the area of electrostatics [7]. The numerical results with $n = 3$ for this example are shown in Table 1 and Figure 1. Here, we use the Newton's method as a non-linear solver with tolerance $1e - 10$. The code were written in Matlab and run on a personal computer (Intel Core2 CPU T5600 @ 1.83Ghz).

Table 1: Numerical approximation for $x(s)$ in Example 2.1 with $n = 3$

s	$x(s)$			Newton's
	Exact	Approx.	Abs. Error	Iteration
0.0	1.00000	0.99880	1.20410×10^{-3}	4
0.1	1.01100	1.00967	1.32537×10^{-3}	4
0.2	1.04800	1.04653	1.46766×10^{-3}	4
0.3	1.11700	1.11543	1.57037×10^{-3}	4
0.4	1.22400	1.22241	1.59206×10^{-3}	4
0.5	1.37500	1.37348	1.51591×10^{-3}	4
0.6	1.57600	1.57465	1.34778×10^{-3}	4
0.7	1.83300	1.83185	1.14943×10^{-3}	4
0.8	2.15200	2.15090	1.09970×10^{-3}	4
0.9	2.53900	2.53797	1.02533×10^{-3}	4
1.0	3.00000	2.99929	7.11181×10^{-4}	4

3 Modified Taylor-Series Method:

In this section, the Taylor series expansion method in Section 1 is applied to Hammerstein equations by first transforming them into equivalent non-linear form which allows each term in the Taylor series to appear under

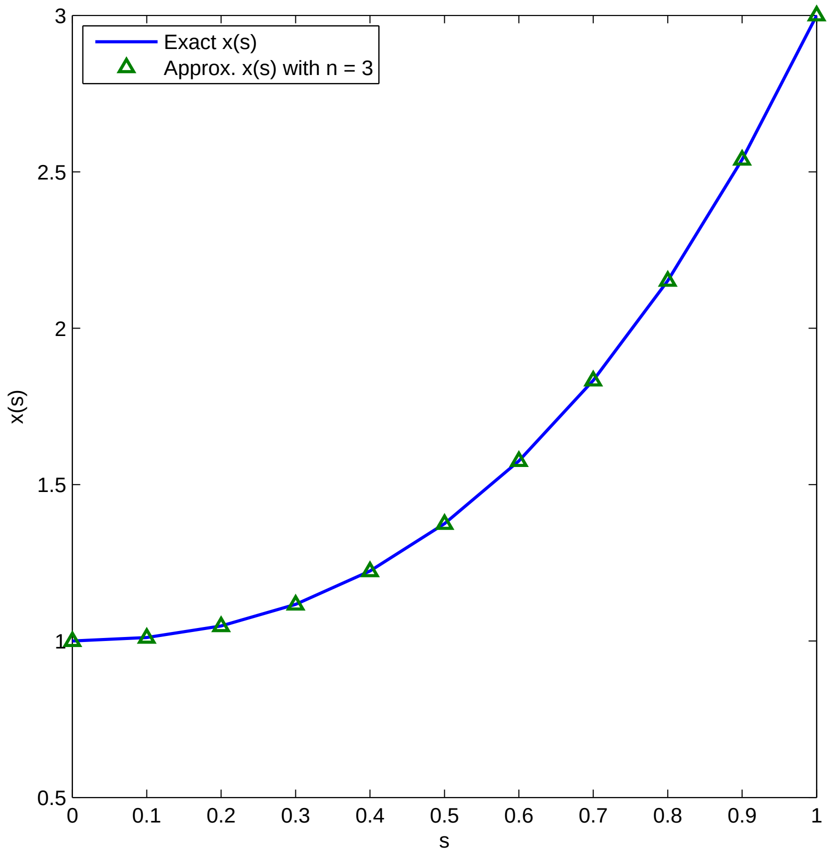

Figure 1: The numerical result from Example 2.1 using a Taylor-series method with $n = 3$.

the linear integral operator. This enables computations of the integrals of the form $\int_0^1 k_s^{(i)}(s,t)\frac{(t-s)^j}{j!}dt$ to be done only once throughout the solution process, thereby speeding up the time required to approximate the solution and reducing the computational complexity. First, we let

$$z(t) \equiv \psi(t, x(t)). \tag{3.1}$$

Equation (1.1) transforms to

$$x(s) - \int_0^1 k(s,t)z(t)dt = y(s), \tag{3.2}$$

for $0 \leq s \leq 1$. Using (3.1) and (3.2),

$$z(s) = \psi(s, y(s) - \int_0^1 k(s,t)z(t)dt), \quad 0 \leq s \leq 1. \tag{3.3}$$

Transformations done in (3.1)-(3.3) were first proposed in [6]. Now,

$$z(t) \approx z(s) + z'(s)(t-s) + \cdots + \frac{1}{n!}z^{(n)}(s)(t-s)^n, \tag{3.4}$$

and substituting this into (3.3), one obtains

$$\begin{aligned} z(s) \approx \psi(s, \quad & y(s) - \textstyle\int_0^1 k(s,t)dt\, z(s) - \int_0^1 k(s,t)(t-s)dt\, z'(s) \\ & - \cdots - \tfrac{1}{n!}\textstyle\int_0^1 k(s,t)(t-s)^n dt\, z^{(n)}(s)) \end{aligned} \tag{3.5}$$

As in (2.2), equation (3.5) represents an nth order nonlinear differential equation which can be solved if n boundary conditions are given. Here, we differentiate (3.3) n times to get, for $i = 1, \ldots, n$,

$$z^{(i)}(s) = \frac{\partial^{(i)}}{\partial s^{(i)}}\psi(s, y(s) - \int_0^1 k(s,t)z(t)dt), \quad 0 \leq s \leq 1. \tag{3.6}$$

and replacing $z(t)$ by (3.4), we obtain

$$\begin{aligned} z^{(i)}(s) \approx \tfrac{\partial^{(i)}}{\partial s^{(i)}}\psi(s, \quad & y(s) - \textstyle\int_0^1 k(s,t)dt\, z(s) - \int_0^1 k(s,t)(t-s)dt\, z'(s) \\ & - \cdots - \tfrac{1}{n!}\textstyle\int_0^1 k(s,t)(t-s)^n dt\, z^{(n)}(s)) \end{aligned} \tag{3.7}$$

Equations (3.5) and (3.7) demonstrate the advantage of transformation (3.1). The unknowns $z(s), z'(s), \ldots, z^{(n)}(s)$ are now placed outside the integrals so that the integrals of the form $\frac{1}{r!}\int_0^1 k_s^{(i)}(s,t)(t-s)^r dt$, $0 \leq r, i \leq n$, are computed once throughout iterations of a nonlinear solver such as the Newton's method.

To make the notation concise, we let

$$\begin{aligned} f_i(z, z', \ldots \quad & , z^{(n)}) \equiv z^{(i)}(s) - \tfrac{\partial^{(i)}}{\partial s^{(i)}}\psi(s, y(s) - \textstyle\int_0^1 k(s,t)dt\, z(s) \\ & - \textstyle\int_0^1 k(s,t)(t-s)dt\, z'(s) - \cdots - \tfrac{1}{n!}\int_0^1 k(s,t)(t-s)^n dt\, z^{(n)}(s)) \end{aligned}$$

for $i = 0, 1, \ldots, n$.

With

$$F(z(s), z'(s), \ldots, z^{(n)}(s)) \equiv \begin{bmatrix} f_0(z(s), z'(s), \ldots, z^{(n)}(s)) \\ f_1(z(s), z'(s), \ldots, z^{(n)}(s)) \\ \vdots \\ f_n(z(s), z'(s), \ldots, z^{(n)}(s)) \end{bmatrix}$$

we find the solutions $\bar{z}(s), \bar{z}'(s), \ldots, \bar{z}^{(n)}(s)$ that satisfy

$$F(\bar{z}(s), \bar{z}'(s), \ldots, \bar{z}^{(n)}(s)) = 0.$$

Once $\bar{z}(s), \bar{z}'(s), \ldots, \bar{z}^{(n)}(s)$ are found, use (3.2) to find an approximation to the solution $x(s)$ of (1.1). It is important once again to note that (3.6) and (3.7) are solved for each s and the solutions $\bar{z}(s), \bar{z}'(s), \ldots, \bar{z}^{(n)}(s)$ at a particular s does not affect the solutions at a different s. Therefore, the solution process can be done in parallel. This is particularly important when dealing with multivariate integral equations which we discuss in our future paper.

An error analysis of modified Taylor series expansion method can be done using the Lipschitz condition on ψ described in condition **(C)** and it is similar to the error analysis done in the previous section. We now present three examples using the modified Taylor series expansion method.

Example 3.1: We consider

$$x(s) - \int_0^1 (s+t)x^2(t)dt = \frac{2}{3}s - \frac{1}{4}, \qquad 0 \le s \le 1.$$

The exact solution is $x(s) = s$. Now, $z(s) = \psi(s, x(s)) = x^2(s)$ and equation (3.3) takes the following form in this particular example;

$$z(s) = \left[\frac{2}{3}s - \frac{1}{4} + \int_0^1 (s+t)z(t)dt\right]^2 \tag{3.8}$$

Substituting

$$z(t) \approx z(s) + z'(s)(t-s) + \frac{1}{2}z''(s)(t-s)^2 \tag{3.9}$$

into (3.8), we obtain

$$\begin{aligned} z(s) = \Bigg[\frac{2}{3}s - \frac{1}{4} + \quad & z(s)\int_0^1 (s+t)dt + z'(s)\int_0^1 (s+t)(t-s)dt \\ & + \frac{z''(s)}{2}\int_0^1 (s+t)(t-s)^2 dt\Bigg]^2 \end{aligned} \tag{3.10}$$

Differentiating (3.8) twice, we obtain

$$z'(s) = 2\left[\frac{2}{3}s - \frac{1}{4} + \int_0^1 (s+t)z(t)dt\right] \cdot \left[\frac{2}{3} + \int_0^1 z(t)dt\right] \tag{3.11}$$

and

$$z''(s) = \left[\frac{2}{3} + \int_0^1 z(t)dt\right]^2 \tag{3.12}$$

Substituting (3.9) into (3.11) and (3.12), two additional equations are obtained. These equations along with (3.10) can be used to solve for $z(s), z'(s)$ and $z''(s)$. It is found that $z(s) = s^2$ and consequently using

$$x(s) = \frac{2}{3}s - \frac{1}{4} + \int_0^1 (s+t)x^2(t)dt,$$

we obtain the exact solution $x(s) = s$ up to the machine accuracy. The numerical results with $n = 2$ are shown in Table 2 and Figure 2.

Table 2: Numerical approximation for $x(s)$ in Example 3.1 with $n = 2$

s	$z(s)$			$x(s)$			Newton's
	Exact	Approx.	Abs. Error	Exact	Approx.	Abs. Error	Iteration
0.0	0.00000	-0.00000	9.91352×10^{-12}	0.00000	0.00000	5.36430×10^{-11}	5
0.1	0.01000	0.01000	9.18050×10^{-14}	0.10000	0.10000	9.32163×10^{-12}	5
0.2	0.04000	0.04000	2.35312×10^{-13}	0.20000	0.20000	1.42539×10^{-12}	5
0.3	0.09000	0.09000	6.74738×10^{-14}	0.30000	0.30000	1.98341×10^{-13}	5
0.4	0.16000	0.16000	1.37113×10^{-14}	0.40000	0.40000	2.65898×10^{-14}	5
0.5	0.25000	0.25000	2.66454×10^{-15}	0.50000	0.50000	3.77476×10^{-15}	5
0.6	0.36000	0.36000	4.44089×10^{-16}	0.60000	0.60000	5.55112×10^{-16}	5
0.7	0.49000	0.49000	3.88578×10^{-16}	0.70000	0.70000	3.33067×10^{-16}	5
0.8	0.64000	0.64000	6.66134×10^{-16}	0.80000	0.80000	5.55112×10^{-16}	5
0.9	0.81000	0.81000	3.33067×10^{-16}	0.90000	0.90000	2.22045×10^{-16}	5
1.0	1.00000	1.00000	1.11022×10^{-16}	1.00000	1.00000	0.00000×10^{0}	5

Example 3.2: Here we consider

$$x(s) - \int_0^1 x^2(t)dt = e^s - \frac{1}{2}(e^2 - 1), \tag{3.13}$$

with $k(s,t) \equiv 1$ and $\psi(s, x(s)) = x^2(s)$ so that $x(s) = e^s$ is the exact solution. Equation (2.3) takes the following form:

$$z(s) = \left[e^s - \frac{1}{2}(e^2 - 1) + \int_0^1 z(t)dt\right]^2.$$

Arguing as in Example 3.1, we obtain the following results. The numerical result with $n = 12$ for this example is shown in Table 3. And, the graph of the numerical results with $n = 4, 8$ and 12 are shown in Figure 3.

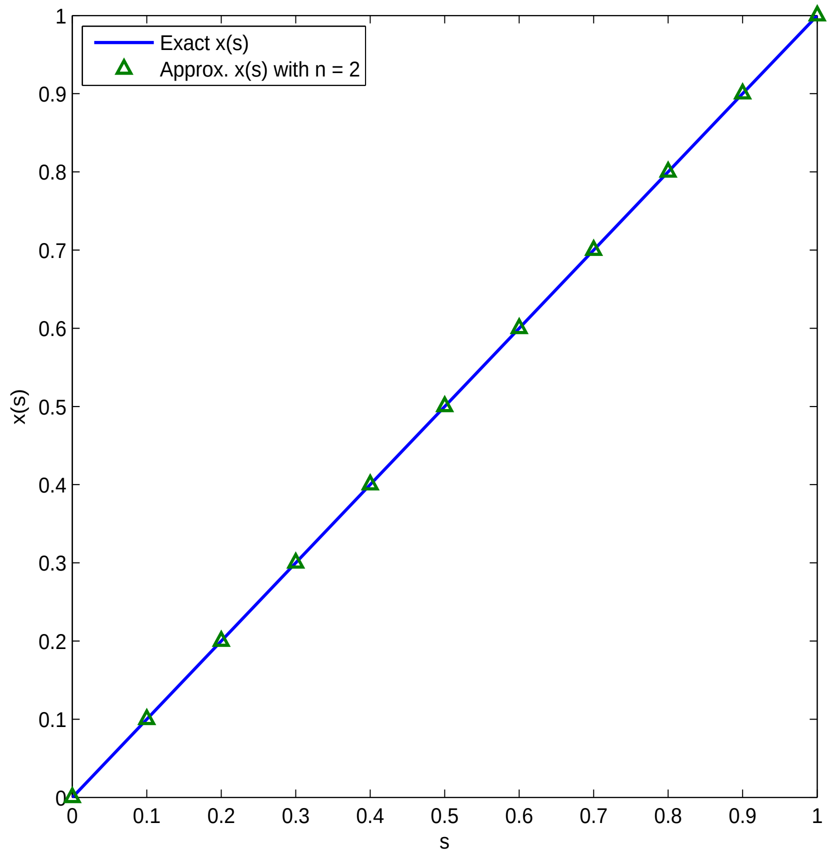

Figure 2: The numerical result from Example 3.1 using a modified Taylor-series method with $n = 2$.

Example 3.3: Here we consider

$$x(s) - \frac{2}{\pi}\int_0^1 \frac{1}{4+(s-t)^2} e^{x(t)} dt = y(s) \tag{3.14}$$

and assume that y is selected so that $x(s) = \sin s$ is the solution. This example is similar to Example 2.1, but nonlinear term and solution are different. The numerical result for this example with $n = 4$ is shown in Table 4. And, the graph of the numerical results with $n = 2$ and 4 are shown in Figure 4.

Table 3: Numerical approximation for $x(s)$ in Example 3.2 with $n = 12$

s	$z(s)$			$x(s)$			Newton's
	Exact	Approx.	Abs. Error	Exact	Approx.	Abs. Error	Iteration
0.0	1.00000	1.00000	8.28526×10^{-8}	1.00000	1.00000	4.88823×10^{-6}	5
0.1	1.22140	1.22140	2.03202×10^{-8}	1.10517	1.10517	1.66629×10^{-6}	5
0.2	1.49182	1.49182	4.19286×10^{-9}	1.22140	1.22140	4.94014×10^{-7}	4
0.3	1.82212	1.82212	1.50170×10^{-8}	1.34986	1.34986	1.28783×10^{-7}	4
0.4	2.22554	2.22554	9.80880×10^{-8}	1.49182	1.49182	1.24572×10^{-7}	4
0.5	2.71828	2.71828	5.09805×10^{-7}	1.64872	1.64872	5.20679×10^{-7}	5
0.6	3.32012	3.32012	7.79238×10^{-8}	1.82212	1.82212	1.12624×10^{-7}	5
0.7	4.05520	4.05520	3.67041×10^{-8}	2.01375	2.01375	1.96428×10^{-7}	5
0.8	4.95303	4.95303	8.21571×10^{-8}	2.22554	2.22554	1.24280×10^{-6}	6
0.9	6.04965	6.04965	2.83171×10^{-7}	2.45960	2.45961	6.22427×10^{-6}	6
1.0	7.38906	7.38905	1.41904×10^{-6}	2.71828	2.71831	2.64610×10^{-5}	6

4 Discrete Taylor-Expansion Method:

In this section, we discuss briefly an idea of discrete Taylor-expansion method. In applying Taylor-expansion method, it is necessary to find the derivatives of the kernel k of Hammerstein equation. The derivatives were computed analytically in examples discussed in the previous sections. Dealing with integral equations with more complicated kernels, these derivatives must be approximated by use of quadratures. More specifically, all the derivatives in equations (2.4) must be approximated. Suppose that we approximate $k_s^{(i)}(s,t)$ by

$$Q^i(k^{(i)}(s,t)) = \sum_{j=0}^{m_i} a_j^i k(s_j^i, t), \tag{4.1}$$

where a_i^j's are weights of the quadrature and s_i^j's are prescribed points which depend upon s, $0 \le i \le n$ and $0 \le j \le m_i$. Then equations in (2.3) become

$$\begin{aligned} x'(s) - \textstyle\int_0^1 \sum_{j=0}^{m_1} a_j^1 k(s_j^1,t)\psi(t, x(s) + x'(s)(t-s) + \frac{x^{(n)}(s)}{n!}(t-s)^n)dt \quad &\approx y'(s), \\ &\vdots \\ x^{(n)}(s) - \textstyle\int_0^1 \sum_{j=0}^{m_n} a_j^n k(s_j^n,t)\psi(t, x(s) + x'(s)(t-s) + \frac{x^{(n)}(s)}{n!}(t-s)^n)dt \quad &\approx y^{(n)}(s). \end{aligned} \tag{4.2}$$

We found that the use of the following central difference formulae (Table 5) for quadrature in (4.1) for interior points s and the forward and backward

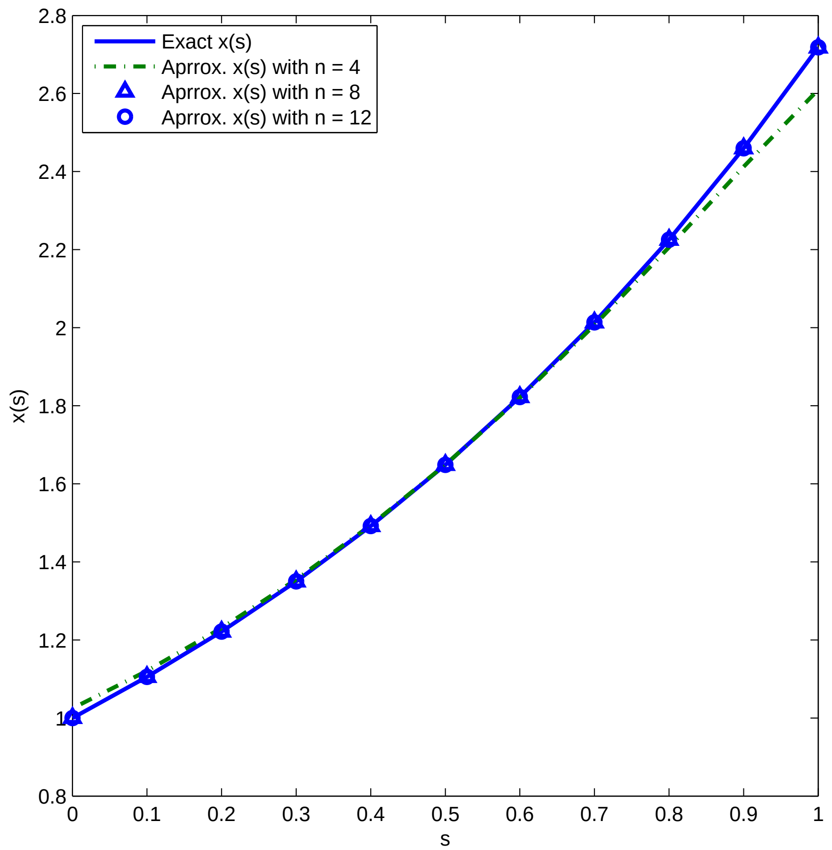

Figure 3: The numerical results from Example 3.2 using a modified Taylor-seried mehtod with $n = 4, 8$ and 12.

formulae (Tables 6 and 7) for the end points $s = 0$ and $s = 1$ provides excellent approximation.

Example 4.1: Here we rework Example 2.1 using the discrete Taylor-expansion method. Recall

$$x(s) - \frac{2}{\pi} \int_0^1 \frac{1}{4 + (s-t)^2} \sin x(t) dt = y(s)$$

and assume that y is selected so that $x(s) = 1 + s^2 + s^3$ is the solution. Quadrature schemes in tables 5, 6 and 7 were used to obtain an approximation. We note that the order of truncation error of all the quadratures used

Table 4: Numerical approximation for $x(s)$ in Example 3.3 with $n = 4$

s	$z(s)$			$x(s)$			Newton's
	Exact	Approx.	Abs. Error	Exact	Approx.	Abs. Error	Iteration
0.0	1.00000	1.00180	1.79872×10^{-3}	0.00000	0.00180	1.79710×10^{-3}	11
0.1	1.10499	1.10594	9.48838×10^{-4}	0.09983	0.10069	8.58319×10^{-4}	11
0.2	1.21978	1.22017	3.95105×10^{-4}	0.19867	0.19899	3.23863×10^{-4}	10
0.3	1.34383	1.34392	9.75300×10^{-5}	0.29552	0.29559	7.25738×10^{-5}	9
0.4	1.47612	1.47610	2.24942×10^{-5}	0.38942	0.38940	1.52388×10^{-5}	8
0.5	1.61515	1.61510	4.29222×10^{-5}	0.47943	0.47940	2.65752×10^{-5}	7
0.6	1.75882	1.75885	2.87394×10^{-5}	0.56464	0.56466	1.63400×10^{-5}	8
0.7	1.90450	1.90487	3.69469×10^{-4}	0.64422	0.64441	1.93979×10^{-4}	10
0.8	2.04901	2.05051	1.50405×10^{-3}	0.71736	0.71809	7.33766×10^{-4}	11
0.9	2.18874	2.19322	4.47973×10^{-3}	0.78333	0.78537	2.04462×10^{-3}	13
1.0	2.31978	2.33071	1.09311×10^{-2}	0.84147	0.84617	4.70108×10^{-3}	14

Table 5: Central difference formulas of order $O(h^2)$

$$f'(x) \approx \frac{f(x+h) - f(x-h)}{2h}$$

$$f''(x) \approx \frac{f(x+h) - 2f(x) + f(x-h)}{h^2}$$

$$f^{(3)}(x) \approx \frac{f(x+2h) - 2f(x+h) + 2f(x-h) - f(x-2h)}{2h^3}$$

$$f^{(4)}(x) \approx \frac{f(x+2h) - 4f(x+h) + 6f(x) - 4f(x-h) + f(x-2h)}{h^4}$$

$$f^{(5)}(x) \approx \frac{f(x+3h) - 4f(x+2h) + 5f(x+h) - 5f(x-h) + 4f(x-2h) - f(x-3h)}{2h^5}$$

$$f^{(6)}(x) \approx \frac{f(x+3h) - 6f(x+2h) + 15f(x+h) - 20f(x) + 15f(x-h) - 6f(x-2h) + f(x-3h)}{h^6}$$

is $O(h^2)$. The numerical results, when compared with the results reported earlier in Example 2.1, are consistent with the order of truncation error. The numerical result with $n = 3$ using discrete Taylor-expansion method is shown in Table 8. We used $h = 0.01$ for the finite difference scheme in our computations.

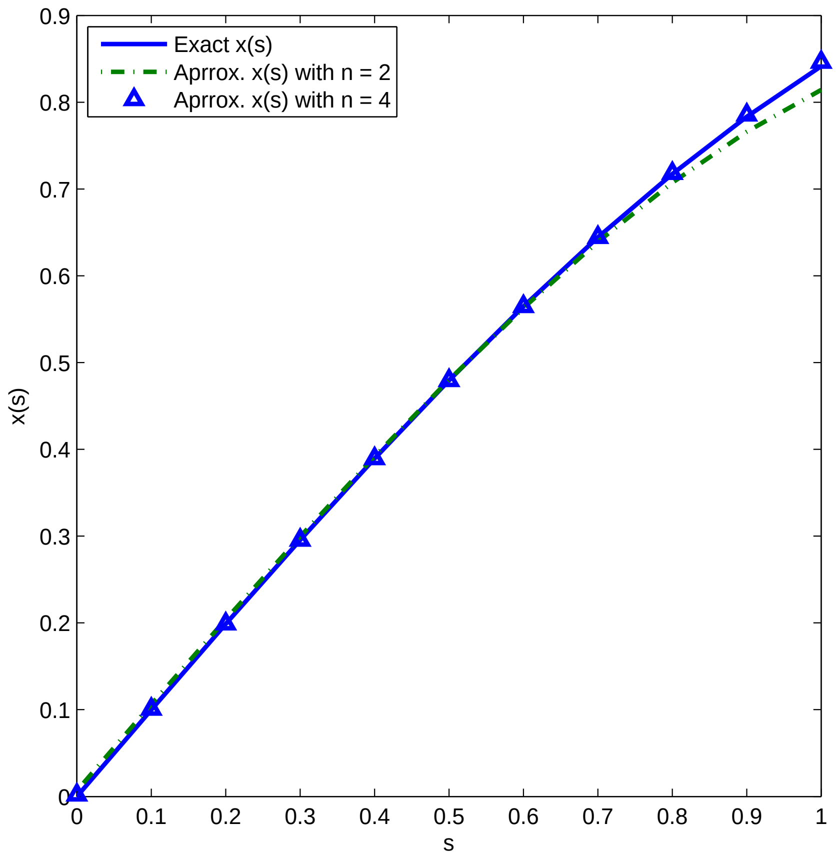

Figure 4: The numerical result from Example 3.3 using a modified Taylor-series method with $n = 2$ and $n = 4$.

References

[1] Yuhe Ren, Bo Zhang, Hong Qiao: A simple Taylor-series expansion method for a class of second integral equations, *Jl. Comp. and Appl. Math.,* Vol. 110, pp. 15-24, (1999).

[2] P. Huabsomboon, B. Novaprateep, H. Kaneko: On Taylor-series expansion method for the second kind integral equations, Accepted for publication in *Jl. Comp. and Appl. Math.*

[3] H. Kaneko and Y. Xu, Superconvergence of the Iterated Galerkin Method for Hammerstein Equations, *S.I.A.M. Journal of Numerical*

Table 6: Forward difference formulas of order $O(h^2)$

$f'(x) \approx \frac{-f(x+2h)+4f(x+h)-3f(x)}{2h}$
$f''(x) \approx \frac{-f(x+3h)+4f(x+2h)-5f(x+h)+2f(x)}{h^2}$
$f^{(3)}(x) \approx \frac{-3f(x+4h)+14f(x+3h)-24f(x+2h)+18f(x+h)-5f(x)}{2h^3}$
$f^{(4)}(x) \approx \frac{-2f(x+5h)+11f(x+4h)-24f(x+3h)+26f(x+2h)-14f(x+h)+3f(x)}{h^4}$
$f^{(5)}(x) \approx \frac{-5f(x+6h)+32f(x+5h)-85f(x+4h)+120f(x+3h)-95f(x+2h)+40f(x+h)-7f(x)}{2h^5}$
$f^{(6)}(x) \approx \frac{-3f(x+7h)+22f(x+6h)-69f(x+5h)+120f(x+4h)-125f(x+3h)+78f(x+2h)-27f(x+h)+4f(x)}{h^6}$

Table 7: Backward difference formulas of order $O(h^2)$

$f'(x) \approx \frac{3f(x)-4f(x-h)+f(x-2h)}{2h}$
$f''(x) \approx \frac{2f(x)-5f(x-h)+4f(x-2h)-f(x-3h)}{h^2}$
$f^{(3)}(x) \approx \frac{5f(x)-18f(x-h)+24f(x-2h)-14f(x-3h)+3f(x-4h)}{2h^3}$
$f^{(4)}(x) \approx \frac{3f(x)-14f(x-h)+26f(x-2h)-24f(x-3h)+11f(x-4h)-2f(x-5h)}{h^4}$
$f^{(5)}(x) \approx \frac{7f(x)-40f(x-h)+95f(x-2h)-120f(x-3h)+85f(x-4h)-32f(x-5h)+5f(x-6h)}{2h^5}$
$f^{(6)}(x) \approx \frac{4f(x)-27f(x-h)+78f(x-2h)-125f(x-3h)+120f(x-4h)-69f(x-5h)+22f(x-6h)-3f(x-7h)}{h^6}$

Analysis, Vol. 33, 1048-1064, (1996).

[4] H. Kaneko and Y. Xu, Degenerate Kernel Method for Hammerstein Equations, *Mathematics of Computation*, Vol. 56, 141-148, (1991).

[5] H. Kaneko, R. Noren, B. Novaprateep: Wavelet applications to the Pertov-Galerkin method for Hammerstein equations, *Appl. Num. Anal.,* Vol. 45, pp. 255-273, (2003).

[6] S. Kumar and I. H. Sloan, A new collocation-type method for Hammerstein equations, *Math. Comp.*, 48(1987), pp 585-593.

[7] E. R. Love, The electrostatic field of two equal circular co-axial conducting discs, *Quart. J. Mech. Appl. Math.,* 2(1949), pp 428-451.

Table 8: Numerical approximation for $x(s)$ in Example 4.1 (using discrete taylor expansion) with $n = 3$

s	$x(s)$			Newton's
	Exact	Approx.	Abs. Error	Iteration
0.0	1.00000	0.99880	1.20410×10^{-3}	4
0.1	1.01100	1.00967	1.32537×10^{-3}	4
0.2	1.04800	1.04653	1.46766×10^{-3}	4
0.3	1.11700	1.11543	1.57020×10^{-3}	4
0.4	1.22400	1.22241	1.59141×10^{-3}	4
0.5	1.37500	1.37349	1.51402×10^{-3}	4
0.6	1.57600	1.57465	1.34778×10^{-3}	4
0.7	1.83300	1.83186	1.14353×10^{-3}	4
0.8	2.15200	2.15090	1.09972×10^{-3}	4
0.9	2.53900	2.53797	1.02533×10^{-3}	4
1.0	3.00000	2.99929	7.10212×10^{-4}	4