ON A CONJECTURE OF S. REICH

Peter Z. Daffer
Department of Mathematics
Macon College
Macon, Georgia 31297

Hideaki Kaneko and **Wu Li**
Department of Mathematics and Statistics
Old Dominion University
Norfolk, Virginia 23529-0077

Abstract

Simeon Reich [6] proved that the fixed point theorem for single-valued mappings proved by Boyd and Wong can be generalized to multivalued mappings which map points into compact sets. He then asked [7] whether his theorem can be extended to multivalued mappings whose range consists of bounded closed sets. In this note, we provide an affirmative answer for a certain subclass of Boyd-Wong contractive mappings.

Mathematics Subject Classification (1991): 47H10

1 Introduction

Let (X, d) be a complete metric space. Denote by $CB(X)$ and $K(X)$ the space of all nonempty closed and bounded subsets of X and the space of all nonempty compact subsets of X, respectively. The Hausdorff metric induced by d will be denoted by H. In [5] (p.40), Reich proved that a mapping $T\colon X \to K(X)$ has a fixed point in X if it satisfies $H(Tx, Ty) \le k(d(x,y))d(x,y)$ for all $x, y \in X$ with $x \ne y$, where $k\colon (0,\infty) \to [0,1)$ satisfies $\limsup_{r\to t^+} k(r) < 1$ for every $t \in (0,\infty)$. This result generalizes the fixed point theorem for single-valued mappings that was proved by Boyd and Wong [1]. One of the conjectures made by Reich in [6],[7] asks whether or not the range of T can be relaxed. Specifically the question is whether or not the range of T, $K(X)$, can be replaced by $CB(X)$. In response to Reich's conjecture, the following theorem was recently proved by Mizoguchi and Takahashi [4], and other proofs have been given by Daffer and Kaneko [3] and Tong-Huei Chang [2];

Theorem 1.1 *Let* (X, d) *be a complete metric space and* $T: X \to CB(X)$. *Assume that* T *satisfies*

$$H(Tx, Ty) \leq k(d(x, y))d(x, y) \tag{1.1}$$

for all $x, y \in X$ *with* $x \neq y$, *where* $k: (0, \infty) \to [0, 1)$ *satisfies* $\limsup_{r \to t^+} k(r) < 1$ *for every* $t \in [0, \infty)$. *Then* T *has a fixed point in* X.

The stronger condition assumed on k in Theorem 1.1, viz., $\limsup_{r \to t^+} k(r) < 1$ for every $t \in [0, \infty)$, implies that $k(t) < h$ for some $0 < h < 1$ and for small $t > 0$. Therefore with this condition, one may conclude that a mapping satisfying (1.1) is a contraction in the sense of Banach over a region for which $d(x, y)$ is sufficiently small. We obtain a theorem which replaces the closed interval $[0, \infty)$ of this result with the open interval $(0, \infty)$ of the classical Boyd-Wong fixed point theorem.

2 The Main Result

In recent papers ([2],[8]) the following class of functions was introduced and studied.

Definition 2.1: Let $\phi: R_+ \to R_+$. ϕ is said to satisfy the condition (Φ) (denoted $\phi \in (\Phi)$) if (i) $\phi(t) < t$ for all $t \in (0, \infty)$; (ii) ϕ is upper semicontinuous from the right on $(0, \infty)$; and (iii) there exists a positive real number s such that ϕ is nondecreasing on $(0, s]$ and $\sum_{n=0}^{\infty} \phi^n(t) < \infty$ for all $t \in (0, s]$.

Chang [2] observed that if $k: (0, \infty) \to [0, 1)$ satisfies $\limsup_{r \to t^+} k(r) < 1$ for every $t \in [0, \infty)$, then there exists a function $\phi \in (\Phi)$ such that $k(t)t \leq \phi(t)$ for all $t \in (0, \infty)$.

Subsequently, Chang proved the following theorem that generalizes Theorem 1.1 above.

Theorem 2.1 *Let* (X, d) *be a complete metric space. Let* $T: X \to CB(X)$ *and suppose that there exists a function* $\phi \in (\Phi)$ *such that*

$$H(Tx, Ty) \leq \phi(\max\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(y, Tx)}{2}\})$$

for all $x, y \in X$. *Then* T *has a fixed point in* X.

The purpose of this note is to establish a class of functions that satisfy $\limsup_{r \to t^+} k(r) < 1$ for every $t \in (0, \infty)$ and that belong to (Φ). Utilizing the result of Chang above, we can then obtain a fixed point theorem for multivalued functions that satisfy the conditions required in the conjecture of Reich. The existence of such a class of functions confirms that the conjecture of Reich is still open and that a further investigation toward a complete resolution is required.

Lemma 2.2 *If $\varphi(t) = t - at^b$, where $a > 0$, then*

(i) if $b \geq 2$, then $\sum_{n=0}^{\infty} \varphi^n(t) = +\infty$, for all $t \in R$ except for $t = 0$ and $t = a^{-\frac{1}{b-1}}$;

(ii) if $1 < b < 2$, then

$$\sum_{n=0}^{\infty} \varphi^n(t) = \begin{cases} < +\infty, & \text{for } 0 \leq t \leq a^{-\frac{1}{b-1}}, \\ = +\infty, & \text{for } t < 0 \text{ or } a^{-\frac{1}{b-1}} < t \end{cases}$$

Here $\varphi^n = \varphi \circ \cdots \circ \varphi$ denotes the n-fold composition.

Proof: A sketch shows that $\sum_{n=0}^{\infty} \varphi^n(t) = +\infty$ for $t < 0$ or $a^{-\frac{1}{b-1}} < t$, for any value of b greater than 1. Also, it is clear that the series converges for $t = 0$ and $t = a^{-\frac{1}{b-1}}$. We thus restrict attention in what follows to $t \in (0, a^{-\frac{1}{b-1}})$.

Let $b \geq 2$ and choose $k > a$, $k \in N$. Then $\frac{1}{n} - \frac{a}{n^b} \geq \frac{1}{n} - \frac{a}{n^2} \geq \frac{1}{n+k}$, for all $n \geq \frac{ak}{k-a} = n_0$. Thus $\varphi(\frac{1}{n}) \geq \frac{1}{n+k}$ for $n \geq n_0$, for some n_0. On the interval $(0, (ab)^{-\frac{1}{b-1}})$, φ is increasing, and so for t in this interval and $N > \max\{[t^{-1}] + 1, n_0\}$ we have $\varphi(t) \geq \frac{1}{n+k}$. It follows that $\varphi^n(t) \geq \frac{1}{N+kn}$. It follows that $\varphi^n(t) \geq \frac{1}{N+kn}$. Indeed, this is true for $n = n_0$, and we see that, for $n \geq n_0$,

$$\varphi^{n+1}(t) = \varphi(\varphi^n(t)) \geq \varphi(\frac{1}{N+kn}) \geq \frac{1}{N+kn+k} = \frac{1}{N+k(n+1)},$$

completing the induction. Hence,

$$\sum_{n=0}^{\infty} \varphi^n(t) \geq \sum_{n=N}^{\infty} \varphi^n(t) \geq \sum_{n=N}^{\infty} \frac{1}{N+kn} = +\infty.$$

Now let $1 < b < 2$. For $1 < c < \frac{1}{b-1}$, we have

$$n^{bc}(\frac{1}{n^c} - \frac{1}{(n+1)^c}) = \frac{n^{bc}((n+1)^c - n^c)}{n^c(n+1)^c} = \frac{n^{bc-c-1}}{(1+\frac{1}{n})^c} \cdot \frac{(1+\frac{1}{n}) - 1}{\frac{1}{n}} \to 0$$

as $n \to \infty$. Thus, there is n_0, which we can choose so that $n_0^{-c} < (ab)^{-\frac{1}{b-1}}$ (so that $\frac{1}{n_0^{-c}}$ is in the interval where φ is increasing) such that $\frac{1}{n^c} - \frac{1}{(n+1)^c} \leq \frac{1}{n^{bc}}$, for all $n \geq n_0$. Thus we have $\varphi(\frac{1}{n^c}) \leq \frac{1}{(n+1)^c}$ for all $n \geq n_0$. For every $t \in [0, a^{-\frac{1}{b-1}}]$, $\lim_{n\to\infty} \varphi^n(t) = 0$, and so for some $k_0 \in N$ we have $\varphi^{k_0} \leq \frac{1}{n_0^c}$. This implies that $\varphi^n(t) \leq \frac{1}{(n_0+n-k_0)^c}$, for all $n \geq k_0$. Indeed, this is true for $n = k_0$, and we have that,

$$\varphi^{n+1}(t) = \varphi(\varphi^n(t)) \leq \varphi(\frac{1}{(n_0+n-k_0+1)^c}) \leq \frac{1}{(n_0+n-k_0+1)^c},$$

completing the induction. Hence

$$\sum_{n=0}^{\infty} \varphi^n(t) = \sum_{n=0}^{k_0-1} \varphi^n(t) + \sum_{n=k_0}^{\infty} \varphi^n(t) = \sum_{n=0}^{k_0-1} \varphi^n(t) + \sum_{n=k_0}^{\infty} \frac{1}{n_0 + n - k_0)^c} < +\infty$$

since $c > 1$. □

Theorem 2.3 *Let (X,d) be a complete metric space and $T: X \to CB(X)$. If there exists a function $\varphi: R_+ \to R_+$ such that $\varphi(t) < t$ for all $t > 0$ and $\varphi(t) \leq t - at^b$, $a > 0$, for some $1 < b < 2$ on some interval $[0,s]$, $s > 0$, such that*

$$H(Tx,Ty) \leq \varphi(d(x,y))$$

for all $x, y \in X$, then T has a fixed point in X.

Proof: Writing $\psi(t) = t - at^b$, we have that $\varphi^n(t) \leq \psi^n(t)$ for every $n \in N$ and $t \in [0,(ab)^{\frac{1}{b-1}}]$ an interval upon which ψ is increasing). Indeed, this holds for $n = 0, 1$ and if it holds for n, then (since $\psi^n(t) \in [0,(ab)^{\frac{1}{b-1}}]$ for every n,

$$\psi^{n+1}(t) = \psi(\psi^n(t)) \geq \psi(\varphi^n(t)) \geq \varphi(\varphi^n(t)) = \varphi^{n+1}(t)$$

completing the induction. By Lemma 2.2, $\psi \in (\Phi)$; hence we also have $\varphi \in (\Phi)$ and the conclusion follows from Theorem 2.1. □

We remark that Theorem 2.3 can be formulated using the contractive condition

$$H(Tx,Ty) \leq \varphi(\max\{d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Ty) + d(y,Tx)}{2}\})$$

because of Theorem 2.1 above. However, because of a more relevance to the conjecture of Reich, we chose to describe our theorem using the contractive condition $H(Tx,Ty) \leq \varphi(d(x,y))$.

Theorem 2.3 reveals that with $k(t) = 1 - at^{b-1}$ we have a class of functions that satisfy all the conditions in the conjecture of Reich. Hence we obtain the following;

Theorem 2.4 *Let (X,d) be a complete metric space and $T: X \to CB(X)$. Assume that T satisfies*

$$H(Tx,Ty) \leq k(d(x,y))d(x,y)$$

for all $x, y \in X$ where $k: R_+ \to [0,1]$ with $k(t) < 1$ for $t > 0$, and $k(t) \leq 1 - at^{b-1}$, $a > 0$, for some $b \in (1,2)$ on some interval $[0,s]$, $s > 0$, $0 < s < a^{-\frac{1}{b-1}}$. Then T has a fixed point in X.

References

[1] D. W. Boyd and J. S. W. Wong, *On Nonlinear Contractions,* Proc. A.M.S. 89(1968), 458-464.

[2] T. H. Chang, *Common Fixed Point Theorems for Multi-Valued Mappings,* Math. Japonica (in press).

[3] P. Z. Daffer and H. Kaneko, *Fixed Points of Generalized Contractive Multi-Valued Mappings,* Jl. Math. Anal. Appl. (in press).

[4] N. Mizoguchi and W. Takahashi, *Fixed Points Theorems for Multi-Valued Mappings on Complete Metric Spaces,* Jl. Math. Anal. Appl. 141(1989) 177-188.

[5] S. Reich, *Fixed Points of Contractive Functions,* Boll. Un. Mat. Ital., (4) 5 (1972), 26-42.

[6] S. Reich, *Some Fixed Point Problems,* Atti. Acad. Naz. Lincei, 57(1974), 194-198.

[7] S. Reich, *Some Problems and Results in Fixed Point Theory,* Contemporay Math. A.M.S. 21(1983), 179-187.

[8] K. Sastry, S.Naidu and J. Prasad, *Common Fixed Points for Mutimaps in a Metric Space,* Nonlinear Anal. Thy. Methods & Appl. 13(3) (1989) 221-229.