Characterizations of Fixed Points of Multi-Valued Maps by Metric Projections

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Abstract

The purpose of this paper is twofold. First a theorem from [2] is generalized by assuming compactness of a certain set of metric projection. Second, we examine a theorem of Dugundji and Granas [5] concerning fixed points of weakly contractive point-valued maps. We prove that the theorem of Dugundji and Granas can be extended to multi-valued maps provided that the corresponding metric projections satisfy the weak contractive condition.

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1 Introduction

Let (X, d) be a metric space, f a continuous map of X into X and T a map of X into $2^X \setminus \{\emptyset\}$. A subset K of X is called **proximinal** if, for each $x \in X$, there exists an element $k \in K$ such that $d(x, k) = d(x, K)$ where $d(x, K) = \inf\{d(x, y) : y \in K\}$. The family of all bounded proximinal subsets of X is denoted by $P(X)$. We denote the family of all nonempty closed subsets of X and that of all nonempty closed and bounded subsets of X by $F(X)$ and $CB(X)$ respectively. A map $\phi: X \times X \to [0, \infty)$ is called **compactly positive** if $\inf\{\phi(x, y) : a \leq d(x, y) \leq b\} > 0$ for each finite interval $[a, b] \subseteq (0, \infty)$. A multi-valued map $T: X \to F(X)$ is called **weakly**

contractive if there exists a compactly positive mapping ϕ such that

$$H(T(x), T(y)) \leq d(x, y) - \phi(x, y)$$

for each $x, y \in X$ where H denotes the Hausdorff metric on $CB(X)$ induced by d. $T\colon X \to F(X)$ is called an **k f-contraction** if there exists $k \in [0, 1)$ such that

$$H(Tx, Ty) \leq kd(fx, fy)$$

for each $x, y \in X$. Moreover, $T\colon X \to F(X)$ is called **f-contractive** if

$$H(Tx, Ty) < d(fx, fy) \qquad \text{whenever}\, fx \neq fy$$

for each $x, y \in X$. A point $x \in X$ is called a **coincidence** point of f and T if $fx \in Tx$. Of course, a k f-contraction generalizes the classical Banach contraction, whereas the idea of f-contractive maps generalizes that of contractive multifunctions that was studied by Smithson in [9]. Recently, it was demonstrated in [3] that fixed point theorem for k f-contractions can be used to prove the existence of solutions of certain classes of nonlinear integral equations. The study of various conditions that guarantee the existence of fixed points of multi-valued k f-contractions and those of f-contractive maps were made in [6,2]. A coincidence point plays a critical role in proving the existence of fixed points of such maps, (see e.g. theorem 1 [2]). Anf**-orbit** of $x \in X$ under T is a sequence $\{x_n : x_n \in X \,\text{and}\, fx_n \in Tx_{n-1}, x_0 = x\}$. An f-orbit of x under T is called **regular** provided that for each n, $d(fx_n, fx_{n+1}) \leq H(Tx_{n-1}, Tx_n)$ and $d(fx_n, fx_{n+1}) \leq d(fx_{n-1}, fx_n)$. An f-orbit of x under T is called **strongly regular** if $T\colon X \to P(X)$ and for each n, $d(fx_{n+1}, fx_n) = d(fx_n, Tx_n)$. In a recent paper [2], the following theorem was proved.

Theorem 1.1 *[2] Let X be a connected metric space, $T\colon X \to F(X)$ an f-contractive map, where $f\colon X \to X$ is continuous. Suppose that $T(X) \subseteq f(X)$ and that $f \circ T(x) \subseteq T \circ f(x)$ for every $x \in X$. If for some $x \in X$ an f-orbit of x is regular and contains a subsequence $\{x_{n_k}\}$ such that $fx_{n_k} \to t_0$ and $fx_{n_k+1} \to t_1$, then $t_0 = t_1$ and $ft_0 \in Tt_0$.*

Our first task in this paper is to re-examine Theorem 1.1. We shall develop a general topological framework that can be used to prove a similar theorem concerning a coincidence point of f and T. The connectivity of X is dropped from assumptions. This will be done in Section 2.

Our second task is to re-examine a theorem of Dugundji and Granas. In [5], they proved that a single-valued weakly contractive map of a complete metric space into itself has a unique fixed point. It is still an open question whether the theorem of Dugundji and Granas can be extended to multi-valued maps. Partial results were provided in theorem 2 [7] and in theorem 2.3 [4]. In Section 3, we prove a theorem that guarantees a fixed point of multi-valued maps provided that their metric projections are weakly contractive.

2 Fixed Points of f-Contractions

Throughout this section, for A_n, $A \in F(X)$, $A_n \Rightarrow A$ denotes the convergence of A_n to A in the Hausdorff metric. For each $x \in X$, we denote the metric projection of x onto A by $P_A(x)$, namely $P_A(x) = \{a \in A : d(x,a) = d(x,A)\}$. We also let $A^{(\epsilon)} = \{x \in X : d(x,A) < \epsilon\}$ and $U_\epsilon(x) = \{y : d(y,x) < \epsilon\}$. We are now ready to prove a sequence of lemmas that will be used to establish the main result of this section.

Lemma 2.1 *Let C_n and $C \in F(X)$. Moreover assume that C is compact. If $C_n \subseteq C^{(\epsilon)}$ for all sufficiently large n and for each $\epsilon > 0$, and if $x_n \in C_n$ for each $n \in N$ (=natural numbers), then there exists a subsequence $\{x_{n_k}\}$ such that $x_{n_k} \to x$ for some $x \in C$ as $k \to \infty$.*

Proof: By passing to a subsequence which we again denote by $\{C_n\}$, we can assume that $C_n \subseteq C^{(\frac{1}{n})}$ for all n. If no subsequence of $\{x_n\}$ converges to any point in C, then to each $x \in C$, there is $\epsilon = \epsilon(x)$ such that $U_\epsilon(x) \cap \{x_n\} = \emptyset$. Now $\{U_{\epsilon(x)}(x) : x \in C\}$ is an open cover of C. Since C is compact, there is a finite subcover $\{U_{\epsilon(x_i)}(x_i) : i = 1, \ldots, N\}$. For large enough n, we have $C^{(\frac{1}{n})} \subseteq \cup_{i=1}^N U_{\epsilon(x_i)}(x_i)$. Moreover, since $\{x_n\} \cap \cup_{i=1}^N U_{\epsilon(x_i)}(x_i) = \emptyset$, we have $C^{(\frac{1}{n})} \cap \{x_n\} = \emptyset$. This contradiction proves the Lemma. $\square$

Now let $A \in P(X)$, $x \in X$, $C = P_A(x)$ and $\widetilde{C}^{(\epsilon)} = \{z \in A : d(z,x) < d(x,C) + \epsilon\}$ for $\epsilon > 0$. Also we denote the complement of A by A^c.

Lemma 2.2 $C = \cap_{\epsilon>0} \widetilde{C}^{(\epsilon)}$.

Proof: It is clear that $C \subseteq \cap_{\epsilon>0} \widetilde{C}^{(\epsilon)}$. If $z \in C^c \cap A$, then for some $\epsilon > 0$, $d(z,x) \geq d(x,C) + \epsilon$. Thus $z \in (\cap_{\epsilon>0} \widetilde{C}^{(\epsilon)})^c = \cup_{\epsilon>0} (\widetilde{C}^{(\epsilon)})^c$. $\square$

Lemma 2.3 $\inf_{z \in (\widetilde{C}^{(\epsilon)})^c \cap A} d(z,x) > d(x,C)$.

Proof: If this is not a case, then there is a sequence $\{z_n\}$, $z_n \in (\widetilde{C}^{(\epsilon)})^c \cap A$ for each n such that $d(x,C) \leq d(z_n,x) < d(x,C) + \frac{1}{n}$. This yields that $z_n \in \widetilde{C}^{(\epsilon)}$ whenever $n > [\frac{1}{\epsilon}]$ which is a contradiction. $\square$

Lemma 2.4 *Let $x_n, x \in X$ and $A_n, A \in P(X)$. Define $C_n = P_{A_n}(x_n)$ and $C = P_A(x)$. Suppose that $x_n \to x$ and $A_n \Rightarrow A$. Then for every $\epsilon > 0$, we have $C_n \subseteq C^{(\epsilon)}$ for all sufficiently large n.*

Proof: If the conclusion is not true, then for some $\epsilon > 0$, there is a subsequence $\{C_{n_k}\}$ and $z_{n_k} \in C_{n_k} \cap (C^{(\epsilon)})^c$ for each k. Now $d(z_{n_k}, x_{n_k}) = d(C_{n_k}, x_{n_k}) = d(A_{n_k}, x_{n_k}) \to d(A, x) = d(C, x)$. Since $x_{n_k} \to x$,

$$\begin{aligned} \limsup_{k\to\infty} d(z_{n_k}, x) &\leq \limsup_{k\to\infty} d(z_{n_k}, x_{n_k}) + \limsup_{k\to\infty} d(x_{n_k}, x) \\ &= d(x, C). \end{aligned} \tag{1}$$

Also since $z_{n_k} \in C_{n_k} \subseteq A_{n_k}$ and $A_{n_k} \Rightarrow A$, we have

$$d(z_{n_k}, (C^{(\epsilon)})^c \cap A) \to 0. \tag{2}$$

Since $C = \cap_{\epsilon>0} C^{(\epsilon)}$ (C is closed) and by Lemma 2.2, $C = \cap_{\epsilon>0} \widetilde{C}^{(\epsilon)}$, using $\widetilde{C}^{(\epsilon)} \subseteq A$, we have that for some ϵ', $0 < \epsilon' \leq \epsilon$, $\widetilde{C}^{(\epsilon')} \subseteq C^{(\epsilon)}$. Hence $z_{n_k} \notin \widetilde{C}^{(\epsilon')}$. There are two cases for z_{n_k}. If $z_{n_k} \in A$, then this implies that

$$d(z_{n_k}, x) \geq d(x, C) + \epsilon' \tag{3}$$

by definition of $\widetilde{C}^{(\epsilon')}$. If $z_{n_k} \notin A$, then $d(z_{n_k}, A) < \frac{\epsilon'}{2}$ for sufficiently large k so that by (2), for k large enough, there is $z' \in A \cap (C^{(\epsilon)})^c$ with $d(z_{n_k}, z') < \frac{\epsilon'}{2}$. In this case we have, since $z' \notin \widetilde{C}^{(\epsilon')}$,

$$\begin{aligned} d(x, C) + \epsilon' &\leq d(z', x) \\ &\leq d(z', z_{n_k}) + d(z_{n_k}, x) \\ &< \frac{\epsilon'}{2} + d(z_{n_k}, x), \end{aligned}$$

i.e.,

$$d(x, C) + \frac{\epsilon'}{2} < d(z_{n_k}, x). \tag{4}$$

In either case, (3) or (4), we get using (1)

$$d(x, C) + \frac{\epsilon'}{2} < \limsup_{k\to\infty} d(z_{n_k}, x) \leq d(x, C).$$

This contradiction shows that there is no subsequence and for every $\epsilon > 0$, $C_n \subseteq C^{(\epsilon)}$ for all sufficiently large n. $\square$

We are now ready to prove our first theorem. The connectivity of metric space (X, d) is dropped from Theorem 1.1 in exchange for an assumption that a certain metric projection is compact.

Theorem 2.5 *Let (X, d) be a metric space and $T: X \to P(X)$ a continuous f-contractive map with $T(X) \subseteq f(X)$. Let f be continuous and one-to-one with f^{-1} continuous on $f(X)$. If for some $x \in X$, a strongly regular f-orbit of x under T has a cluster point x^*, and if $P_{Tx^*}(fx^*)$ is a compact set, then x^* is a coincidence point of f and T.*

Proof: Let $\{x_n\}$, $fx_{n+1} \in Tx_n$, for each $n \geq 0$, $x_0 = x$, be a strongly regular f-orbit of x under T and put $c_n = d(fx_{n+1}, fx_n)$. If $c_n = 0$ for some n, then $0 = d(fx_{n+1}, fx_n) = d(fx_n, Tx_n)$ by the strong regularity of the f-orbit. Hence x_n is a coincidence point of f and T and the proof is complete. We thus assume that $c_n > 0$ for all n. First notice that $\{c_n\}$ is monotonically strictly decreasing. To see this, $c_{n+1} = d(fx_{n+2}, fx_{n+1}) = d(Tx_{n+1}, fx_{n+1}) \leq H(Tx_{n+1}, Tx_n) < d(fx_{n+1}, fx_n) = c_n$. Thus we assume that $\lim_{n\to\infty} c_n = c$ for some $c \geq 0$.

By hypothesis there is a subsequence $\{x_{n_k}\}$ of the f-orbit that converges to x^*. By the continuity of T and f we have

$$\begin{aligned} c = \lim_{k\to\infty} c_{n_k} &= \lim_{k\to\infty} d(fx_{n_k+1}, fx_{n_k}) \\ &= \lim_{k\to\infty} d(Tx_{n_k}, fx_{n_k}) \\ &= d(Tx^*, fx^*). \end{aligned}$$

We finish the proof by showing that $c = 0$. Suppose $c > 0$. Let $C \equiv P_{Tx^*}(fx^*)$. Then C is compact by hypothesis. For every $z \in C$, we have $c = d(Tx^*, fx^*) = d(z, fx^*)$. Recall that $T(X) \subseteq f(X)$. Since f^{-1} is continuous, $f^{-1}(C)$ is compact. For every $y \in f^{-1}(C)$, we have $c = d(Tx^*, fx^*) = d(fy, fx^*) > H(Ty, Tx^*)$. Let $\varphi(y) \equiv H(Ty, Tx^*)$. Then $\varphi: f^{-1}(C) \to R_+$ is continuous and since $f^{-1}(C)$ is compact, $\sup_{y\in f^{-1}(C)} \varphi(y) \equiv r < d(fy, fx^*) = c$.

Now put $C_k \equiv P_{Tx_{n_k}}(fx_{n_k})$. Then by the strong regularity of the f-orbit, $fx_{n_k+1} \in C_k$. Using the continuity of T and f, we have $fx_{n_k} \to fx^*$ and $Tx_{n_k} \to Tx^*$. By Lemma 4, for every $\epsilon > 0$, $C_k \subseteq C^{(\epsilon)}$ for all sufficiently large k. Now $fx_{n_k+1} \in C_k$ for every k. Lemma 1 yields a subsequence of $\{fx_{n_k+1}\}$, which we denote by the same notation, which converges to some point $z \in C$. Then since f is continuous and one-to-one, $x_{n_k+1} = f^{-1}fx_{n_k+1} \to f^{-1}z \equiv y \in f^{-1}(C)$. It now follows that

$$\lim_{k\to\infty} H(Tx_{n_k+1}, Tx_{n_k}) = H(Ty, Tx^*) \leq r < c. \tag{5}$$

But by the strong regularity of the f-orbit,

$$H(Tx_{n_k+1}, Tx_{n_k}) \geq d(Tx_{n_k+1}, fx_{n_k+1}) = d(fx_{n_k+1}, fx_{n_k+2}) = c_{n_k+2}.$$

Since $\lim_{n\to\infty} c_n = c$, $\lim_{k\to\infty} c_{n_k+2} = c$ and hence $\liminf_{k\to\infty} H(Tx_{n_k+1}, Tx_{n_k}) \geq c$, which contradicts (5). Thus we conclude that $c = 0$ and this yields $d(Tx^*, fx^*) = 0$, implying $fx^* \in Tx^*$, i.e., x^* is a coincidence point. $\square$

Theorem 2.6 *Suppose all the hypotheses in Theorem 2.5 are satisfied. Suppose further that* $f \circ T(x) \subseteq T \circ f(x)$ *for every coincidence point* x^* *of* f *and* T*. If* $\{f^n x^*\}$ *converges, then* f *and* T *possess a common fixed point .*

Proof: Since $f \circ T \subseteq T \circ f$ for every coincidence point of f and T, we have $f^n \circ Tx^* \subseteq T \circ f^n x^*$. If $f^n x^* \to x$, we get from $fx^* \in Tx^*$ that $f^{n+1}x^* \in T \circ f^n x^*$ and letting $n \to \infty$, we obtain $x \in Tx$, i.e., x is a fixed point of T. □

REMARK 1: If (X, d) is a connected metric space, then an f contractive map $T: X \to F(X)$ is continuous [2] and this need not be assumed.
REMARK 2: If $T: X \to \breve{C}(X)$ or if $T: X \to K(X)$, where $\breve{C}(X)$ stands for the Chebyshev subsets [8] and $K(X)$ the nonempty compact subsets of X, then $P_{Tx^*}(fx^*)$ is compact and this assumption can be dropped.

3 Fixed Points of Weak Contractions

Throughout this section, we assume that (X, d) is a complete metric space. In this section, we study metric projections that are weakly contractive. We let $\delta(A, B) \equiv \sup\{d(x, y): x \in A, y \in B\}$ for A, $B \subseteq X$. The following lemma generalizes lemma 1.3 of Dugundji and Granas [5].

Lemma 3.1 *Let* $T: X \to P(X)$ *satisfy*

$$H(P_{Tx}(x), P_{Ty}(y)) \leq d(x, y) - \varphi(x, y), \qquad \text{for all } x, y \in X$$

where $\varphi: X \times X \to R_+$ *is compactly positive. For* $r > 0$*, write* $\Lambda(\frac{r}{2}, r) = \inf\{\varphi(x, y): \frac{r}{2} \leq d(x, y) < r\}$*. Let* $x_0 \in X$*. If* $d(x_0, P_{Tx_0}(x_0)) < \min\{\frac{r}{2}, \Lambda(\frac{r}{2}, r)\}$*, then* $P_{Ty}(y) \cap U_r(x_0) \neq \emptyset$ *for every* $y \in U_r(x_0)$*.*

Proof: Let $y \in U_r(x_0)$. Then

$$\begin{aligned} d(x_0, P_{Ty}(y)) &\leq d(x_0, P_{Tx_0}(x_0)) + H(P_{Tx_0}(x_0), P_{Ty}(y)) \\ &< \min\{\frac{r}{2}, \Lambda(\frac{r}{2}, r)\} + d(x_0, y) - \varphi(x_0, y). \end{aligned}$$

Two cases arise: (i) $d(x_0, y) < \frac{r}{2}$. Here $d(x_0, P_{Ty}(y)) < \frac{r}{2} + \frac{r}{2} = r$. (ii) $\frac{r}{2} \leq d(x_0, y) \leq r$. Here $d(x_0, P_{Ty}(y)) < \Lambda(\frac{r}{2}, r) + r - \Lambda(\frac{r}{2}, r) = r$. Thus in all cases $d(x_0, P_{Ty}(y)) < r$. Hence there exists $z \in P_{Ty}(y)$ with $d(x_0, z) < r$. □

Theorem 3.2 *Let (X,d) be a complete metric space and $T\colon X \to P(X)$ be continuous. Suppose that T satisfies*

$$H(P_{Tx}(x), P_{Ty}(y)) \le d(x,y) - \varphi(x,y) \qquad \text{for all } x,y \in X$$

where φ is compactly positive. Then T has a unique fixed point.

Proof: Let $x_0 \in X$ and let $\{x_n\}$ be a strongly regular orbit of x_0 under T. Then

$$\begin{aligned} d(x_n, x_{n+1}) &= d(x_n, P_{Tx_n}(x_n)) \le H(P_{Tx_{n-1}}(x_{n-1}), P_{Tx_n}(x_n)) \\ &\le d(x_{n-1}, x_n) - \varphi(x_{n-1}, x_n) \\ &\le d(x_{n-2}, x_{n-1}) - \varphi(x_{n-2}, x_{n-1}) - \varphi(x_{n-1}, x_n) \\ &\qquad \cdots \\ &\le d(x_0, x_1) - \sum_{i=0}^{n-1} \varphi(x_i, x_{i+1}). \end{aligned}$$

Thus $\sum_{i=0}^{n-1} \varphi(x_i, x_{i+1}) \le d(x_0, x_1)$ and $\sum_{i=0}^{\infty} \varphi(x_i, x_{i+1}) < \infty$. Write $c(\varphi, r) = \min\{\frac{r}{2}, \Lambda(\frac{r}{2}, r)\}$, with $\Lambda(\frac{r}{2}, r)$ as in Lemma 3.1. To every $k \in N$, there is $n_k \in N$ such that

$$\varphi(x_n, x_{n+1}) < \inf\{\varphi(\xi,\eta)\colon c(\varphi, \frac{1}{k}) \le d(\xi,\eta) \le d(x_0,x_1)\}$$

for all $n \ge n_k$. Hence for $n \ge n_k$, we have

$$d(x_n, x_{n+1}) = d(x_n, P_{Tx_n}(x_n)) < c(\varphi, \frac{1}{k}).$$

We may choose the sequence $\{n_k\}$ to be increasing. By Lemma 3.1, $P_{Tx_{n_k}}(x_{n_k}) \cap U_{\frac{1}{k}}(x_{n_k}) \neq \emptyset$, and so x_{n_k+1} can be chosen in this intersection. Then $x_{n_k+1} \in U_{\frac{1}{k}}(x_{n_k})$ and again by Lemma 3.1 x_{n_k+2} can be chosen in $P_{Tx_{n_k+1}}(x_{n_k+1}) \cap U_{\frac{1}{k}}(x_{n_k}) \neq \emptyset$. Inductively, we get that $x_{n_k+j} \in U_{\frac{1}{k}}(x_{n_k})$ and so x_{n_k+j+1} can be chosen in $P_{Tx_{n_k+j}}(x_{n_k+j}) \cap U_{\frac{1}{k}}(x_{n_k})$ for all $j \ge 0$.

We now replace the original strongly regular orbit $\{x_n\}$ by a new one $\{\tilde{x}_n\}$. For all $n \le n_1$, we put $\tilde{x}_n = x_n$. For $k = 1$ and $n > n_1$, we replace $\{x_n\}$ by a new sequence $\{x_n^{(1)}\}$ formed as described above. We thus have $x_n^{(1)} \in U_1(x_{n_1})$ for all $n \ge n_1$. For $k = 2$ and $n > n_2$, we replace the terms of the sequence $\{x_n^{(1)}\}$ by new ones $\{x_n^{(2)}\}$ using the procedure described above. Thus $x_n^{(2)} \in U_{\frac{1}{2}}(x_{n_2})$ for all $n \ge n_2$. In general, having arrived at $\{x_n^{(k-1)}\}$ for all $n > n_k$ replace $x_n^{(k-1)}$ by $x_n^{(k)} \in U_{\frac{1}{k}}(x_{n_k})$. This generates a new sequence $\{\tilde{x}_n\}$, which is also a strongly regular orbit of x_0. This sequence has the property that $\tilde{x}_n \in U_{\frac{1}{k}}(\tilde{x}_{n_k})$ for all $n \ge n_k$ and for every $k \in N$. We claim that the sequence thus constructed is Cauchy. To prove this claim, let $\epsilon > 0$ and choose $k > \frac{1}{\epsilon}$. Then for all $m, n \ge n_k$ we have $\tilde{x}_m, \tilde{x}_n \in U_{\frac{1}{k}}(\tilde{x}_{n_k})$ and so $d(\tilde{x}_m, \tilde{x}_n) \le \frac{2}{k} < 2\epsilon$. Now since X is complete, $\{\tilde{x}_n\}$ converges, say $\tilde{x}_n \to z$. Since T is continuous, we have $T\tilde{x}_n \to Tz$, and an appeal to lemma 2 of Assad and Kirk [1] now yields $z \in Tz$, i.e., T has a fixed point. Uniqueness of fixed point is clear. □

REMARK 3: In Theorem 3.2, the continuity condition assumed on T was necessitated because of the appeal to lemma 2 of Assad and Kirk [1] that we made at the end of the proof. The following lemma shows that Theorem 3.2 can be improved by relaxing the continuity assumption on T. First we recall the following [8]; $T: X \to CB(X)$ is said to be **upper Kuratowski semi-continuous (u.K.s.c)** at x_0, respectively **lower Kuratowski semi-continuous (l.K.s.c.)** at x_0, if $lim_{n\to\infty} x_n = x_0$, $y_n \in T(x_n)$, $n \in N$, $\lim_{n\to\infty} y_n = y_0$ imply $y_0 \in T(x_0)$, respectively if $\lim_{n\to\infty} x_n = x_0$, $y_0 \in T(x_0)$ imply $\lim d(y_0, T(x_n)) = 0$. $T: X \to CB(X)$ is said to be **upper semi-continuous (u.s.c.)** at x_0, respectively **lower semi-continuous (l.s.c.)** at x_0 if for every open set $U \subset X$ such that $T(x_0) \subset U$, respectively such that $T(x_0) \cap U \neq \emptyset$, there exists in X an open neighborhood V of x_0 such that $T(x) \subset U$ for all $x \in V$, respectively such that $T(x) \cap U \neq \emptyset$ for all $x \in V$. Finally, $T: X \to CB(X)$ is said to be **upper Hausdorff semi-continuous (u.H.s.c.)** at x_0, respectively **lower Hausdorff semi-continuous (l.H.s.c.)** at x_0, if $\lim_{n\to\infty} x_n = x_0$ implies $\lim_{n\to\infty} \sup_{g\in T(x_n)} d(g, T(x_0)) = 0$, respectively $\lim_{n\to\infty} \sup_{g\in T(x_0)} d(g, T(x_n)) = 0$.

Lemma 3.3 *Let (X, d) be a metric space and let $T: X \to CB(X)$ and let $\{x_n\}$ be a sequence such that $x_{n+1} \in T(x_n)$, for $n \in N$. Suppose that $x_n \to x_0$ and that T is u.s.c., u.K.s.c. or u.H.s.c., then $x_0 \in T(x_0)$, i.e., x_0 is a fixed point of T.*

Proof: Let T be u.s.c. and suppose that $x_0 \notin T(x_0)$. Let $0 < \epsilon < \frac{1}{3} d(x_0, T(x_0))$ be given. Define $V \equiv \{y \in X | d(y, T(x_0)) < \epsilon\}$. Since T is u.s.c., there is $\delta > 0$ such that $d(x, x_0) < \delta$ implies $T(x) \subset V$. We choose $\delta < \frac{1}{3} d(x_0, T(x_0))$. Now choose n_0 such that $n \geq n_0$ implies $d(x_n, x_0) < \delta$. Then $T(x_n) \subset V$. But since $x_{n+1} \in T(x_n)$, we have $x_{n+1} \in V$. Hence, for $n \geq n_0 + 1$, we have $d(x_n, x_0) > \frac{1}{3} d(x_0, T(x_0))$. This contradiction proves this lemma.

If T is u.K.s.c., then by the definition of u.K.s.c. the result follows immediately.

If T is u.H.s.c., then $x_n \to x_0$ implies that $\sup_{x\in T(x_n)} d(x, T(x_0)) \to 0$. Since $d(x_0, T(x_0)) \leq d(x_0, x_n) + d(x_n, T(x_0)) \leq d(x_0, x_n) + \sup_{x\in T(x_{n-1})} d(x, T(x_0))$ and since the last two terms tend to 0 as $n \to \infty$, we conclude that $d(x_0, T(x_0)) = 0$ and $x_0 \in T(x_0)$. (We note that this last result in fact takes care of u.s.c. case, since u.s.c. implies u.H.s.c. [8;p.56]) □

Theorem 3.2 can now be improved vastly using the lemma 3.3.

Theorem 3.4 *Let (X, d) be a complete metric space and $T: X \to P(X)$ be u.s.c., u.K.s.c. or* *u.H.s.c. Suppose further that T satisfies*

$$H(P_{Tx}(x), P_{Ty}(y)) \leq d(x,y) - \varphi(x,y) \qquad \textit{for all } x,y \in X$$

where φ is compactly positive. Then T has a unique fixed point.

REMARK 4: Lemma 3.3 does not hold when T is l.s.c., l.K.s.c. or l.H.s.c. This can be seen by the following example. Let $T: R \to CB(R)$ be defined by $T(x) = [0,1]$ for $x \neq 0$ and $T(0) = \{1\}$. Then T is l.s.c., l.K.s.c., and l.H.s.c. But it is not true that if $x_n \to x_0$ and $x_{n+1} \in T(x_n)$, then $x_0 \in T(x_0)$. Let $\{x_n\}$ be a sequence in $(0,1]$ which converges to 0. Then $x_{n+1} \in T(x_n) = [0,1]$ for every n, but $x = 0 \notin T(0) = \{1\}$. Of course this example does not provide a counterexample to Theorem 3.2. It is interesting to investigate whether or not the continuity conditions on T can be further weakened.

REMARK 5: The condition of Theorem 3.2 can be replaced by the following stronger condition.

$$H(P_{Tx}(x), P_{Ty}(y)) \leq \Phi(d(x,y)),$$

where $\Phi: R_+ \to R_+$ satisfies $\Phi(t) < t$ and $\limsup_{s \to t} \Phi(s) < t$, for all $t > 0$. Indeed, defining $\varphi(t) = t - \Phi(t)$, we have $H(P_{Tx}(x), P_{Ty}(y)) \leq d(x,y) - \varphi(d(x,y))$. To see that φ is compactly positive, consider the interval $[a,b]$, $0 < a \leq b < \infty$. If $\inf\{\varphi(s) | s \in [a,b]\} = 0$, then there is a sequence $\{s_n\}$ in $[a,b]$ such that $\lim_{n\to\infty} \varphi(s_n) = 0$. If $s_n \to s$ for some $s \in [a,b]$, then $\lim_{n\to\infty}(s_n - \Phi(s_n)) = s - \lim_{n\to\infty} \Phi(s_n) = 0$ so that $\lim_{n\to\infty} \Phi(s_n) = s$. This contradiction proves that φ is compactly positive.

REMARK 6: It is interesting to ask whether or not the condition $\limsup_{s\to t} \Phi(s) < t$, for all $t > 0$ in REMARK 5 can be replaced by the condition $\limsup_{s\to t^+} \Phi(s) < t$, for all $t > 0$

References

1. N. A. Assad and W. A. Kirk, Fixed point theorems for set-valued mappings of contractive type, *Pacific J. Math.* 43 (1972), 553-562.

2. P. Z. Daffer and H. Kaneko, Multi-Valued f-Contractive Mappings, *Boll. Unione Mat. Ital.* (7) 8-A (1994), 233-241.

3. P. Z. Daffer and H. Kaneko, Applications of f-Contraction Mappings to Nonlinear Integral Equations, *Bull. Inst. Math. Acad. Sinica* 22, (1994), 69-74.

4. P. Z. Daffer and H. Kaneko, Fixed Points of Generalized Contractive Multi-Valued Mappings, *Jl. Math. Anal. and Appl.* (in press).

5. J. Dugundji and A. Granas, Weakly contractive maps and elementary domain invariance theorem,*Bull. Greek Math. Soc.* 19 (1978), 141-151.

6. H. Kaneko, Single-Valued and Multi-Valued f-Contractions, *Boll. Unione Mat. Ital.* (6) 4-A (1985), 29-33.

7. H. Kaneko, Generalized contractive multi-valued mappings and their fixed points, *Math. Japon.* 33 (1988), 57-64.

8. I. Singer, THEORY OF BEST APPROXIMATION AND FUNCTIONAL ANALYSIS, Regional Conference Series in Applied Mathematics, SIAM (1974).

9. R. Smithson, Fixed Points for Contractive Multifunctions, *Proc. Amer. Math. Soc.* 27 (1), (1971), 192-194.