International Journal of Numerical Methods and Applications
Volume 1, Number 2, 2009, Pages 139-153
This paper is available online at http://www.pphmj.com

DISCRETE TAYLOR-EXPANSION METHOD FOR INTEGRAL EQUATIONS

PALLOP HUABSOMBOON, BORIBOON NOVAPRATEEP and HIDEAKI KANEKO*

Department of Mathematics
Faculty of Science
Mahidol University
Bangkok, 10400, Thailand

*Department of Mathematics and Statistics
Old Dominion University
Norfolk, Virginia 23529-0077, U. S. A.

Abstract

In this paper, we continue our study that began in the papers [2], [3] and [4] concerning the Taylor-expansion method for approximating the solution of integral equations. The Taylor-expansion methods require taking the derivatives of the kernel of an integral equation. In practical applications, these derivatives must be approximated due to the complexities of the kernels involved. This paper addresses this issue.

1. Introduction

Recently, in a series of papers [2], [3] and [4], the present authors generalized the idea which was first proposed by Ren et al. [8]. The idea was to use the Taylor-expansion method to approximate the solution of the Fredholm integral equations of the second kind having convolution type kernels. Taylor-series expansion method developed in [8] relies heavily upon the condition that a kernel $k(|t-s|)$ of convolution type

2000 Mathematics Subject Classification: 65R20, 45B05.

Keywords and phrases: Taylor-expansion method, Fredholm integral equation.

Received January 7, 2009

decays rapidly as $|t-s|$ increases. As stated in [4], the method of Ren et al., therefore, does not apply to a wider class of the second kind Fredholm integral equations having kernels of different variations. Also, the accuracy of approximation depends critically upon the rate at which a convolution kernel approaches zero as $|t-s| \to \infty$. The present authors [2], recently generalized the results in [8] to obtain a new Taylor-series expansion method which is applicable to a larger class of Fredholm equations. The new method is also capable of delivering more accurate approximations. Another important characteristic of the new Taylor-expansion method is that the method lends itself to a parallel computation environment. Most techniques used to approximate the solution of an integral equation such as the method of Galerkin, collocation or least squares, calculate the solution over the entire interval. This result, for many cases, in a large system of linear equations in which matrix involved is usually dense and thus expensive to solve its corresponding system. In contrast, the current Taylor-expansion method is capable of computing a solution at a single point s. Also, the size of matrix involved is much smaller than the size of matrices used in the Galerkin, collocation or least-squares method. Finally, the new Taylor-expansion method not only approximates the solution of an integral equation but also approximates the derivatives of the solution concurrently, the feature none of the aforementioned numerical methods can claim to possess.

We note that in [3], the new expansion method was extended to obtain approximation of the solution of nonlinear Hammerstein equation and in [4], the new technique was extended to approximate the solution of a Volterra equation.

The Taylor-expansion method requires taking derivatives of a kernel of an integral equation. In practical applications, these derivatives must be approximated due to the complexities of the kernels involved. This paper addresses this issue.

This paper is organized as follows: The main point of discrete Taylor-expansion method is given in Section 2. A list of quadratures used in numerical experiments and results of the numerical experiments are given in Section 3. In Section 4, we generalize the method to systems of Fredholm integral equations.

2. Discrete Taylor-expansion Method

We begin this section by first outlining the recently established Taylor-expansion method. The method is demonstrated relative to the Fredholm integral equation of the second kind. Fredholm equation of the second kind can be written as

$$x(s) - \int_0^1 k(s, t)x(t)dt = y(s), \quad 0 \le s \le 1, \tag{2.1}$$

where it is assumed that 1 is not the eigenvalue of the operator

$$Tx(s) \equiv \int_0^1 k(s, t)x(t)dt.$$

We assume throughout the paper that $k \in C^n([0, 1] \times [0, 1])$ so that the operator T is a compact linear operator of $C[0, 1]$ into $C[0, 1]$. Fredholm equation of the second kind plays an important role in many physical applications which include potential theory and Dirichlet problems, particle transport problems of astrophysics and radiative heat transfer problems. A reader may consult the references provided in [8] for these applied problems.

The Taylor-expansion method begin by first writing

$$x(t) \approx x(s) + x'(s)(t - s) + \cdots + \frac{1}{n!}x^{(n)}(s)(t - s)^n. \tag{2.2}$$

Substituting (2.2) for $x(t)$ in the integral in (2.1), we obtain

$$\left[1 - \int_0^1 k(s, t)dt\right]x(s) - \left[\int_0^1 k(s, t)(t - s)dt\right]x'(s) - \cdots$$

$$\cdots - \left[\frac{1}{n!}\int_0^1 k(s, t)(t - s)^n dt\right]x^{(n)}(s) \approx y(s), \quad 0 < s < 1. \tag{2.3}$$

This represents an nth order linear ordinary differential equation with variable coefficients, which can be solved if n boundary conditions are known. These boundary conditions are normally supplied from a physical experiment and thus they may not be always readily available. To circumvent the problem, we differentiate (2.1) n times to get

$$\begin{aligned} x'(s) - \int_0^1 k_s'(s, t)x(t)dt &= y'(s) \\ \vdots \qquad\qquad \vdots \qquad\qquad & \quad \vdots \\ x^{(n)}(s) - \int_0^1 k_s^{(n)}(s, t)x(t)dt &= y^{(n)}(s), \end{aligned} \tag{2.4}$$

where $k_s^{(i)}(s, t) = \frac{\partial^i}{\partial s^i} k(s, t)$, $i = 1, \ldots, n$. Now, replace each $x(t)$ in (2.4) by the right side of (2.2) to obtain

$$\begin{aligned} &-\int_0^1 k_s'(s, t)dtx(s) - \left[\int_0^1 k_s'(s, t)(t-s)dt - 1\right]x'(s) - \cdots \\ &\qquad \cdots - \frac{1}{n!}\int_0^1 k_s'(s, t)(t-s)^n dtx^{(n)}(s) \approx y'(s) \\ &\qquad \vdots \qquad\qquad\qquad \vdots \qquad\qquad\qquad \vdots \\ &-\int_0^1 k_s^{(n)}(s, t)dtx(s) - \int_0^1 k_s^{(n)}(s, t)(t-s)dtx'(s) - \cdots \\ &\qquad \cdots - \left[\frac{1}{n!}\int_0^1 k_s^{(n)}(s, t)(t-s)^n dt - 1\right]x^{(n)}(s) \approx y^{(n)}(s). \end{aligned} \tag{2.5}$$

Combining equations (2.3) and (2.5), we obtain the following system of linear equations, the solutions of which we denote by $\bar{x}(s), \bar{x}'(s), \ldots, \bar{x}^{(n)}(s)$ for an arbitrary but fixed $s \in [0, 1]$,

$$\begin{bmatrix} 1 - \int_0^1 k(s, t)dt & -\int_0^1 k(s, t)(t-s)dt & \cdots & -\frac{1}{n!}\int_0^1 k(s, t)(t-s)^n dt \\ -\int_0^1 k_s'(s, t)dt & 1 - \int_0^1 k_s'(s, t)(t-s)dt & \cdots & -\frac{1}{n!}\int_0^1 k_s'(s, t)(t-s)^n dt \\ \vdots & \vdots & \ddots & \vdots \\ -\int_0^1 k_s^{(n)}(s, t)dt & -\int_0^1 k_s^{(n)}(s, t)(t-s)dt & \cdots & 1 - \frac{1}{n!}\int_0^1 k_s^{(n)}(s, t)(t-s)^n dt \end{bmatrix}$$

$$\times \begin{bmatrix} \bar{x}(s) \\ \bar{x}'(s) \\ \vdots \\ \bar{x}^{(n)} \end{bmatrix} = \begin{bmatrix} y(s) \\ y'(s) \\ \vdots \\ y^{(n)}(s) \end{bmatrix}. \tag{2.6}$$

The size n of the matrix in (2.6) depends on the degree of Taylor polynomial used in approximation and our numerical experiments show that the scale of $n = 5$ or 6 is more than enough to provide accurate approximations for most problems. Recall that in a method such as Galerkin or collocation, the resulting size of the matrix is much larger. For example, in the Galerkin method with quadratic spline basis functions applied over 100 equally spaced elements, the size of the matrix is 300×300. However, it should be pointed out that the current method is applicable to integral equations with smooth kernels and therefore, in the case of Fredholm equations with weakly singular kernels, it is recommended that the Galerkin or collocation method is used to obtain accurate solutions. For recent advances in the numerical treatments of linear as well as nonlinear integral equations, see, e.g., [5-7] and references cited within the related literature.

In order to implement the Taylor-expansion method, the components of the matrix in (2.6) must be computed. In practical applications, they must be numerically approximated due to the complexity of the kernel. In this paper, we discuss this issue. Suppose that we approximate $k_s^{(i)}(s, t)$ by

$$Q^i(k^{(i)}(s, t)) = \sum_{j=0}^{m_i} a_j^i k(s_j^i, t), \tag{2.7}$$

where a_i^j's are weights of the quadrature and s_i^j's are prescribed points which depend upon s, $0 \le i \le n$ and $0 \le j \le m$. Then equation (2.6) transforms into the following, the solutions of the system are denoted by $x^*(s), x^{*'}(s), \ldots, x^{*(n)}(s)$

$$\begin{bmatrix} 1-\int_0^1 k(s,t)dt & \cdots & -\frac{1}{n!}\int_0^1 k(s,t)(t-s)^n dt \\ -\int_0^1 \sum_{j=0}^{m_1} a_j^1 k(s_j^1, t)dt & \cdots & -\frac{1}{n!}\int_0^1 \sum_{j=0}^{m_1} a_j^1 k(s_j^1, t)(t-s)^n dt \\ \vdots & \ddots & \vdots \\ -\int_0^1 \sum_{j=0}^{m_n} a_j^n k(s_j^n, t)dt & \cdots & 1-\frac{1}{n!}\int_0^1 \sum_{j=0}^{m_n} a_j^n k(s_j^n, t)(t-s)^n dt \end{bmatrix}$$

$$\times \begin{bmatrix} x^*(s) \\ x^{*'}(s) \\ \vdots \\ x^{*(n)}(s) \end{bmatrix} = \begin{bmatrix} y(s) \\ y'(s) \\ \vdots \\ y^{(n)}(s) \end{bmatrix}. \tag{2.8}$$

In order to construct an error analysis, we denote (2.6) and (2.8), respectively, by

$$A\bar{x} = y, \tag{2.9}$$

and

$$A^* x^* = y. \tag{2.10}$$

Then the following are straightforward and its proofs can be found in any standard textbook of numerical analysis, e.g., see [1]. In what follows $\|\cdot\|$ denotes a vector and its corresponding matrix norm.

Theorem 1. *Suppose A is nonsingular so that equation* (2.9) *has a unique solution. Also, assume that $A^* = A + \delta A^*$ and that*

$$\|\delta A^*\| \le \frac{1}{\|A^{-1}\|}$$

so that A^* is also nonsingular. With $x^* = \bar{x} + \delta x^*$, we have

$$\frac{\|\delta x^*\|}{\|\bar{x}\|} \le \frac{condA}{1 - condA\frac{\|\delta A^*\|}{\|A\|}} \cdot \frac{\|\delta A^*\|}{\|A\|}. \tag{2.11}$$

Error $|x^{(i)}(s) - \bar{x}^{(i)}(s)|$, for each $i = 0, 1, \ldots, n$, can be found as follows. This was established in [2] but we include it here for completeness of analysis. In order to analyze the error term of the current method, substituting $x(t)$ in (2.1) by its Taylor series, we obtain

$$x(s) - \int_0^1 k(s, t)\left[\sum_{r=0}^{n} \frac{x^{(r)}(s)}{r!}(t-s)^r + \frac{x^{(n+1)}(\xi(s))}{(n+1)!}(t-s)^{n+1}\right]dt = f(s), \quad (2.12)$$

for $0 \le s \le 1$. From the first equation of (2.6),

$$\bar{x}(s) - \int_0^1 k(s, t)\sum_{r=0}^{n} \frac{\bar{x}^{(r)}(s)}{r!}(t-s)^r dt = f(s), \quad 0 \le s \le 1. \quad (2.13)$$

From (2.12) and (2.13),

$$[x(s) - \bar{x}(s)] - \int_0^1 k(s, t)\sum_{r=0}^{n} \frac{[x^{(r)}(s) - \bar{x}^{(r)}(s)]}{r!}(t-s)^r$$

$$= \frac{x^{(n+1)}(\xi(s))}{(n+1)!}\int_0^1 k(s, t)(t-s)^{n+1} dt. \quad (2.14)$$

Proceeding similarly for the remaining equations in (2.6), the errors $x^{(r)}(s) - \bar{x}^{(r)}(s)$, $r = 0, 1, \ldots, n$ can be computed by solving

$$A \times \begin{bmatrix} x(s) - \bar{x}(s) \\ \vdots \\ \vdots \\ x^{(n)}(s) - \bar{x}^{(n)}(s) \end{bmatrix} = \begin{bmatrix} \frac{x^{(n+1)}(\xi(s))}{(n+1)!}\int_0^1 k(s, t)(t-s)^{n+1} dt \\ \vdots \\ \vdots \\ \frac{x^{(n+1)}(\xi(s))}{(n+1)!}\int_0^1 k_s^{(n)}(s, t)(t-s)^{n+1} dt \end{bmatrix}. \quad (2.15)$$

Right side of equation (2.15) reveals that the Taylor-expansion method finds a solution exactly if the computation is carried out without round-off errors provided that the solution is a polynomial. It also confirms that the error of the approximation of the Taylor-expansion method converges to zero as $n \to \infty$ under the current assumption of smooth kernel. Denote (2.15) by $AE = F$ so that the vector E of errors can be bounded as

$$\| E \| \leq \| A^{-1} \| \| F \|. \tag{2.16}$$

From (2.11) and (2.15), we obtain the following:

Theorem 2. *For each* $s \in [0, 1]$, *let* $x(s)$ *be the solution of* (2.1) *and* $x = [x(s), x'(s), \ldots, x^{(n)}(s)]^T$. *Also, let* x^* *denote the solution of* (2.10). *Then*

$$\| x - x^* \| \leq \| A^{-1} \| \| F \| + \frac{condA}{1 - condA \dfrac{\| \delta A^* \|}{\| A \|}} \cdot \frac{\| \delta A^* \|}{\| A \|} \| x \|. \tag{2.17}$$

3. Quadratures and Numerical Examples

For demonstration, we use a collection of central difference approximations for $k_s^{(i)}(s, t)$ all of whose local truncation error is $O(h^2)$ (see Table 1). However, the forward or backward difference formulas will be used when $s = 0$ or $s = 1$ (at the lower or upper limit of an integration), respectively. Those difference formulas are given in Tables 2 and 3.

Table 1. Central difference formulas of order $O(h^2)$

$f'(x) \approx \dfrac{f(x+h) - f(x-h)}{2h}$
$f''(x) \approx \dfrac{f(x+h) - 2f(x) + f(x-h)}{h^2}$
$f^{(3)}(x) \approx \dfrac{f(x+2h) - 2f(x+h) + 2f(x-h) - f(x-2h)}{2h^3}$
$f^{(4)}(x) \approx \dfrac{f(x+2h) - 4f(x+h) + 6f(x) - 4f(x-h) + f(x-2h)}{h^4}$
$f^{(5)}(x) \approx \dfrac{f(x+3h) - 4f(x+2h) + 5f(x+h) - 5f(x-h) + 4f(x-2h) - f(x-3h)}{2h^5}$
$f^{(6)}(x) \approx \dfrac{f(x+3h) - 6f(x+2h) + 15f(x+h) - 20f(x) + 15f(x-h) - 6f(x-2h) + f(x-3h)}{h^6}$

Table 2. Forward difference formulas of order $O(h^2)$

$$f'(x) \approx \frac{-f(x+2h)+4f(x+h)-3f(x)}{2h}$$

$$f''(x) \approx \frac{-f(x+3h)+4f(x+2h)-5f(x+h)+2f(x)}{h^2}$$

$$f^{(3)}(x) \approx \frac{-3f(x+4h)+14f(x+3h)-24f(x+2h)+18f(x+h)-5f(x)}{2h^3}$$

$$f^{(4)}(x) \approx \frac{-2f(x+5h)+11f(x+4h)-24f(x+3h)+26f(x+2h)-14f(x+h)+3f(x)}{h^4}$$

$$f^{(5)}(x) \approx \frac{-5f(x+6h)+32f(x+5h)-85f(x+4h)+120f(x+3h)-95f(x+2h)+40f(x+h)-7f(x)}{2h^5}$$

$$f^{(6)}(x) \approx \frac{-3f(x+7h)+22f(x+6h)-69f(x+5h)+120f(x+4h)-125f(x+3h)+78f(x+2h)-27f(x+h)+4f(x)}{h^6}$$

Table 3. Backward difference formulas of order $O(h^2)$

$$f'(x) \approx \frac{3f(x)-4f(x-h)+f(x-2h)}{2h}$$

$$f''(x) \approx \frac{2f(x)-5f(x-h)+4f(x-2h)-f(x-3h)}{h^2}$$

$$f^{(3)}(x) \approx \frac{5f(x)-18f(x-h)+24f(x-2h)-14f(x-3h)+3f(x-4h)}{2h^3}$$

$$f^{(4)}(x) \approx \frac{3f(x)-14f(x-h)+26f(x-2h)-24f(x-3h)+11f(x-4h)-2f(x-5h)}{h^4}$$

$$f^{(5)}(x) \approx \frac{7f(x)-40f(x-h)+95f(x-2h)-120f(x-3h)+85f(x-4h)-32f(x-5h)+5f(x-6h)}{2h^5}$$

$$f^{(6)}(x) \approx \frac{4f(x)-27f(x-h)+78f(x-2h)-125f(x-3h)+120f(x-4h)-69f(x-5h)+22f(x-6h)-3f(x-7h)}{h^6}$$

In this paper, all computations were done using Matlab 7.1 Release 14 on a personal computer (Intel Core2 CPU T5600 @ 1.83GHz). It took less than one minute to get the numerical results. We used $h = 0.01$ for the difference formulas in our computations.

Example 3.1. Consider (Example 2.2 in [2])

$$x(s) - \frac{2}{\pi}\int_0^1 k(s, t)x(t)dt = y(s), \quad 0 \le s \le 1, \tag{3.1}$$

where $k(s, t) = [4 + (s-t)^2]^{-1}$ and $y(s)$ is chosen so that $x(s) = 1 + s^2 + s^5$ is the solution. Numerical results with $n = 6$ gave excellent approximations for x as well as for its derivatives. The numerical approximation for $x(s)$ and its first derivative with $n = 6$ are shown in Table 4. And, the graph of the numerical results with $n = 2, 4$ and 6 is shown in Figure 1.

Table 4. Numerical approximation for $x(s)$ and its first derivative in Example 3.1 with $n = 6$

s	$x(s)$			$x'(s)$		
	Exact	Approx.	Abs. Error	Exact	Approx.	Abs. Error
0.0	1.00000	1.00042	4.21368×10^{-4}	0.00000	0.00014	1.44969×10^{-4}
0.1	1.01001	1.01024	2.29400×10^{-4}	0.20050	0.20057	6.88810×10^{-5}
0.2	1.04032	1.04043	1.13087×10^{-4}	0.40800	0.40803	2.99118×10^{-5}
0.3	1.09243	1.09248	4.90904×10^{-5}	0.64050	0.64051	1.09721×10^{-5}
0.4	1.17024	1.17026	1.68592×10^{-5}	0.92800	0.92800	3.30327×10^{-6}
0.5	1.28125	1.28125	4.57707×10^{-8}	1.31250	1.31250	1.63568×10^{-6}
0.6	1.43776	1.43775	1.30106×10^{-5}	1.84800	1.84800	3.36711×10^{-6}
0.7	1.65807	1.65804	2.90853×10^{-5}	2.60050	2.60051	7.40580×10^{-6}
0.8	1.96768	1.96763	4.96264×10^{-5}	3.64800	3.64801	1.43322×10^{-5}
0.9	2.40049	2.40042	6.98985×10^{-5}	5.08050	5.08052	2.24703×10^{-5}
1.0	3.00000	2.99992	7.80123×10^{-5}	7.00000	7.00002	2.31397×10^{-5}

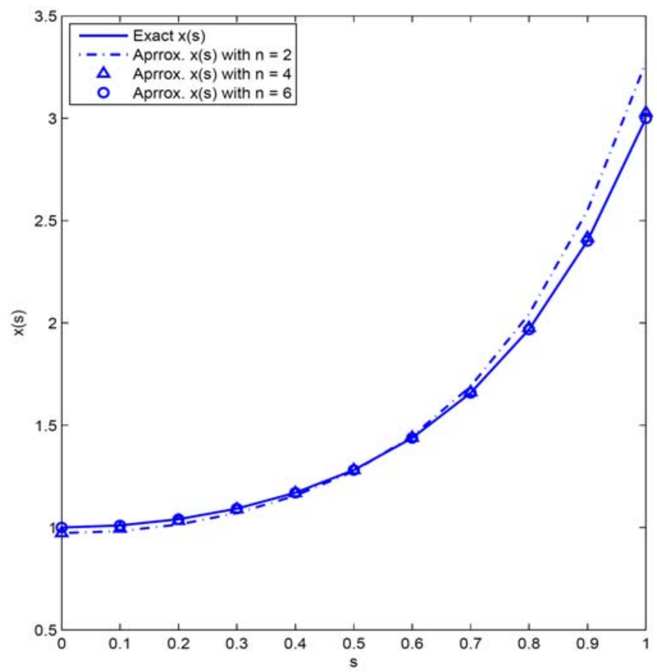

Figure 1. The numerical results from Example 3.1 with $n = 2, 4$ and 6.

Example 3.2. In this example (Example 2.3 in [2]), the kernel is the same as in Example 3.1 but $y(s)$ is chosen so that $x(s) = \exp(2s)$ is the solution. The numerical approximation for $x(s)$ and its first derivative with $n = 6$ are shown in Table 5. And, the graph of the numerical results with $n = 2,\ 4$ and 6 is shown in Figure 2.

Table 5. Numerical approximation for $x(s)$ and its first derivative in Example 3.2 with $n = 6$

s	$x(s)$			$x'(s)$		
	Exact	Approx.	Abs. Error	Exact	Approx.	Abs. Error
0.0	1.00000	1.02964	2.96438×10^{-2}	2.00000	1.88360	1.16404×10^{-1}
0.1	1.22140	1.23974	1.83380×10^{-2}	2.44281	2.33559	1.07211×10^{-1}
0.2	1.49182	1.50304	1.12177×10^{-2}	2.98365	2.88976	9.38868×10^{-2}
0.3	1.82212	1.82930	7.17623×10^{-3}	3.64424	3.56701	7.72300×10^{-2}
0.4	2.22554	2.23073	5.19125×10^{-3}	4.45108	4.39307	5.80106×10^{-2}
0.5	2.71828	2.72265	4.36755×10^{-3}	5.43656	5.39961	3.69492×10^{-2}
0.6	3.32012	3.32412	4.00049×10^{-3}	6.64023	6.62553	1.47037×10^{-2}
0.7	4.05520	4.05884	3.64425×10^{-3}	8.11040	8.11851	8.11164×10^{-3}
0.8	4.95303	4.95620	3.16871×10^{-3}	9.90606	9.93695	3.08820×10^{-2}
0.9	6.04965	6.05245	2.80276×10^{-3}	12.09929	12.15223	5.29338×10^{-2}
1.0	7.38906	7.39229	3.23192×10^{-3}	14.77811	14.85157	7.34585×10^{-2}

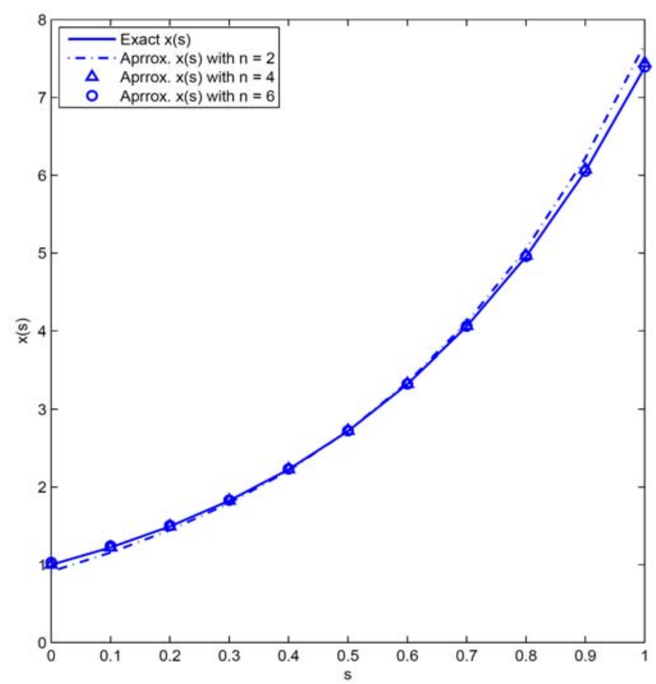

Figure 2. The numerical results from Example 3.2 with $n = 2,\ 4$ and 6.

4. System of Fredholm Integral Equations

In this section, we consider applying the Taylor technique described in the previous section to systems of Fredholm integral equations of the second kind. Consider the system of Fredholm equations of the second kind,

$$\mathbf{X}(s) - \int_0^1 \mathbf{K}(s, t)\mathbf{X}(t)dt = \mathbf{Y}(s), \quad 0 \leq s \leq 1,$$

where

$$\mathbf{X}(s) = [x_1(s), x_2(s), ..., x_m(s)]^T,$$

$$\mathbf{Y}(s) = [y_1(s), y_2(s), ..., y_m(s)]^T,$$

$$\mathbf{K}(s, t) = [k_{ij}(s, t)], \quad i, j = 1, 2, ..., m. \tag{4.1}$$

The ith equation of (3.1) is given by

$$x_i(s) - \int_0^1 \sum_{j=1}^{m} k_{ij}(s, t)x_j(t)dt = y_i(s), \quad i = 1, 2, ..., m. \tag{4.2}$$

Since

$$x_j(t) \approx x_j(s) + x_j'(s)(t-s) + \cdots + \frac{1}{n!}x_j^{(n)}(s)(t-s)^n, \tag{4.3}$$

substituting (4.3) into (4.2), we obtain

$$x_i(s) - \sum_{r=0}^{n}\sum_{j=1}^{m}\frac{1}{r!}\left[\int_0^1 k_{ij}(s, t)(t-s)^r dt\right]x_j^{(r)}(s) \approx y_i(s), \quad i = 1, 2, ..., m. \tag{4.4}$$

Differentiating (4.2) n times, we have

$$x_i^{(l)}(s) - \int_0^1 \sum_{j=1}^{m} k_{ij_s}^{(l)}(s, t)x_j(t)dt = y_i^{(l)}(s), \quad i = 1, 2, ..., m, \quad l = 1, ..., n. \tag{4.5}$$

Substituting once again (4.3), this time into (4.5), we get

$$x_i^{(l)}(s) - \sum_{r=0}^{n} \sum_{j=1}^{m} \frac{1}{r!} \left[\int_0^1 k_{ij_s}^{(l)}(s, t)(t-s)^r dt \right] x_j^{(r)}(s) \approx y_i^{(l)}(s),$$

$$i = 1, 2, \ldots, m, \quad l = 1, \ldots, n. \tag{4.6}$$

Equations (4.4) and (4.6) represent a system of $(n+1)m$ equations in as many unknown functions, $\{x_i^{(l)}(s)\}$, $i = 1, \ldots, m$; $l = 0, 1, \ldots, n$. The current method was tested on the following examples. Note that these examples were tested in [2] in which the necessary derivatives were found analytically. Numerical results below show that the discrete Taylor-expansion method works quite well and produces numerical results which are accurate within the truncation errors of the quadratures.

Example 4.1. In this example, we consider the following Fredholm system of integral equations (Example 3.1 in [2]):

$$\begin{cases} x_1(s) - \int_0^1 (s-t)^3 x_1(t)dt - \int_0^1 (s-t)^2 x_2(t)dt = y_1(s) \\ x_2(s) - \int_0^1 (s-t)^4 x_1(t)dt - \int_0^1 (s-t)^3 x_2(t)dt = y_2(s) \end{cases} \tag{4.7}$$

with $y_1(s) = \frac{1}{20} - \frac{11}{30}s + \frac{5}{3}s^2 - \frac{1}{3}s^3$ and $y_2(s) = -\frac{1}{30} - \frac{41}{60}s + \frac{3}{20}s^2 + \frac{23}{12}s^3 - \frac{1}{3}s^4$. The exact solutions are $x_1(s) = s^2$ and $x_2(s) = -s + s^2 + s^3$. The numerical solutions with $n = 4$ are shown in Table 6.

Example 4.2. Finally, we consider the following Fredholm system of integral equations (Example 3.2 in [2]):

$$\begin{cases} x_1(s) + \int_0^1 t \cos s \, x_1(t)dt + \int_0^1 s \sin t \, x_2(t)dt = y_1(s) \\ x_2(s) + \int_0^1 e^{st^2} x_1(t)dt + \int_0^1 (s+t) x_2(t)dt = y_2(s) \end{cases} \tag{4.8}$$

with $y_1(s) = \frac{\cos s}{3} + \frac{s \sin^2 1}{2} + s$ and $y_2(s) = \frac{e^s - 1}{2s} + \cos s + (s+1)\sin 1$

$+ \cos 1 - 1$. The exact solutions are $x_1(s) = s$ and $x_2(s) = \cos s$. The numerical solutions with $n = 4$ are shown in Table 7.

Table 6. Numerical approximation for $x_1(s)$ and $x_2(s)$ in Example 4.1 with $n = 4$

s	$x_1(s)$			$x_2(s)$		
	Exact	Approx.	Abs. Error	Exact	Approx.	Abs. Error
0.0	0.00000	-0.00001	1.41485×10^{-5}	0.00000	0.00001	1.34590×10^{-5}
0.1	0.01000	0.00999	1.26317×10^{-5}	-0.08900	-0.08899	9.13115×10^{-6}
0.2	0.04000	0.03999	6.53544×10^{-6}	-0.15200	-0.15200	4.22754×10^{-6}
0.3	0.09000	0.09000	2.92529×10^{-6}	-0.18300	-0.18300	1.75259×10^{-6}
0.4	0.16000	0.16000	9.29941×10^{-7}	-0.17600	-0.17600	6.78388×10^{-7}
0.5	0.25000	0.25000	5.22233×10^{-8}	-0.12500	-0.12500	3.24046×10^{-7}
0.6	0.36000	0.36000	3.33121×10^{-7}	-0.02400	-0.02400	2.18999×10^{-7}
0.7	0.49000	0.49000	8.50065×10^{-8}	0.13300	0.13300	4.56240×10^{-8}
0.8	0.64000	0.64000	1.52955×10^{-6}	0.35200	0.35200	9.77705×10^{-7}
0.9	0.81000	0.81000	4.67749×10^{-6}	0.63900	0.63900	3.35674×10^{-6}
1.0	1.00000	0.99996	4.10313×10^{-5}	1.00000	0.99997	3.45666×10^{-5}

Table 7. Numerical approximation for $x_1(s)$ and $x_2(s)$ in Example 3.4 with $n = 4$

s	$x_1(s)$			$x_2(s)$		
	Exact	Approx.	Abs. Error	Exact	Approx.	Abs. Error
0.1	0.10000	0.10001	9.35049×10^{-6}	0.99500	0.99493	7.85780×10^{-5}
0.2	0.20000	0.20000	2.26520×10^{-6}	0.98007	0.98001	5.49130×10^{-5}
0.3	0.30000	0.30000	1.88048×10^{-6}	0.95534	0.95530	3.32472×10^{-5}
0.4	0.40000	0.40000	2.45258×10^{-6}	0.92106	0.92104	1.64162×10^{-5}
0.5	0.50000	0.50000	1.77266×10^{-6}	0.87758	0.87758	4.41430×10^{-6}
0.6	0.60000	0.60000	1.78311×10^{-6}	0.82534	0.82534	9.00345×10^{-6}
0.7	0.70000	0.70000	4.07381×10^{-6}	0.76484	0.76488	3.90125×10^{-5}
0.8	0.80000	0.79999	1.03443×10^{-5}	0.69671	0.69682	1.16283×10^{-4}
0.9	0.90000	0.89998	2.26467×10^{-5}	0.62161	0.62191	2.98192×10^{-4}
1.0	1.00000	0.99996	4.18021×10^{-5}	0.54030	0.54099	6.88327×10^{-4}

References

[1] K. E. Atkinson, An Introduction to Numerical Analysis, 2nd ed., John Wiley and Sons, 1989.

[2] P. Huabsomboon, B. Novaprateep and H. Kaneko, On Taylor-series expansion methods for the second kind integral equations, J. Comput. Appl. Math., submitted.

[3] P. Huabsomboon, B. Novaprateep and H. Kaneko, Taylor-series expansion methods for nonlinear Hammerstein equations, Appl. Math. Comp., to appear.

[4] P. Huabsomboon, B. Novaprateep and H. Kaneko, Taylor-series expansion method for Volterra integral equations of the second kind, Appl. Math. Comp., submitted.

[5] H. Kaneko, R. D. Noren and B. Novaprateep, Wavelet applications to the Petrov-Galerkin method for Hammerstein equations, Appl. Numer. Math. 45 (2003), 255-273.

[6] H. Kaneko and Y. Xu, Degenerate kernel method for Hammerstein equations, Math. Comp. 56 (1991), 141-148.

[7] H. Kaneko and Y. Xu, Superconvergence of the iterated Galerkin methods for Hammerstein equations, SIAM J. Numer. Anal. 33 (1996), 1048-1064.

[8] Yuhe Ren, Bo Zhang and Hong Qiao, A simple Taylor-series expansion method for a class of second kind integral equations, J. Comput. Appl. Math. 110 (1999), 15-24.