Numerical solutions of Hammerstein equations

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Abstract

In this chapter, we survey recent results on the numerical solutions of the Hammerstein equations. Hammerstein equations arise naturally in connection with the Laplace equation with a certain class of nonlinear boundary conditions. The Hammerstein equations with smooth as well as weakly singular kernels will be treated.

1 Introduction

In this chapter, we make a report on some of the recently obtained numerical methods for finding an approximate solution of the Hammerstein equation,

$$x(t) - \int_0^1 k(t,s)\psi(s,x(s))\,ds = f(t), \qquad t \in [0,1], \tag{1}$$

where k,f and ψ are known functions and x is the solution to be determined. In the past two decades, there has been considerable interest in the numerical analysis of solutions of integral equations. A comprehensive survey of numerical methods for the solution of the Fredholm integral equation of the second kind was made by Atkinson [1] in 1976. The survey encompasses most of the standard numerical techniques available to the practitioners and it is a prerequisite for the current article. Since the publication of Atkinson's survey, numerous new research articles have appeared. In particular, many interesting numerical techniques that deal with the weakly singular Fredholm integral equations have been established, e.g.

[2],[3], . Also a substantial number of results obtained for the Fredholm equations were generalized to nonlinear Hammerstein equations. The one-dimensional Hammerstein equation arises naturally in connection with the Laplace equations with nonlinear boundary conditions that are proposed in R^2. The purpose of this survey article is to update readers, mathematicians and engineers alike, on these new advances that have occurred in the area of Hammerstein equations. The majority of the materials presented in this article are taken primarily from the recent collaborative work done by Yuesheng Xu and the present authors. We point out that Vainikko [4] recently wrote a monograph on the numerical methods for multidimensional weakly singular Fredholm integral equations of the second kind. The methods described in Vainikko's monograph are primarily based upon an approximation using irregular grids which correspond to the materials presented in Section 6 of the present work. We feel that many of the results in [5] can be extended to multidimensional weakly singular Hammerstein equations. We also feel that most of the methods described in this article in relation to one dimensional equations can be extended to hold for multidimensional Hammerstein equations.

This article consists of two parts. The first part, which is comprised of Sections 2, 3 and 4, is concerned with Hammerstein equations with kernels that are smooth. We begin this part in Section 2 by discussing the degenerate kernel method. The degenerate kernel method is a classical technique for approximating the solutions of Fredholm equations. This well known method was generalized to Hammerstein equations in [5]. In Section 3, the collocation and Galerkin methods for Hammerstein equations will be presented. The topic of the superconvergence of the iterates of the numerical solutions of these methods is taken up in Section 4. The second part is devoted to the numerical analysis of the weakly singular Hammerstein equations. When dealing with the weakly singular equations, one must be concerned with the nature of the regularity of the solution, since, without this knowledge, no numerical method would be successful. To achieve an optimal convergence rate of numerical solutions, it is imperative that we have a proper understanding of the kind of singularity that a solution possesses. We discuss this issue in Section 5. When solving numerically the weakly singular equations, one is required to evaluate a large number of weakly singular integrals. In the event of weakly singular kernel of convolution type, the technique of product-integration appears to have been a popular choice to approximate such integrals. The work of Piessens and Branders [6] and those of Sloan [7] and of Sloan and Smith [8] should be mentioned in this context. The critical recursion formulae in their product-integration methods, however, do not hold when integrands are altered slightly. Kaneko and Xu [9], on the other hand, established the Gauss-type numerical quadratures that can be applied to a wider variety of weakly singular integrals. A review of these quadrature schemes is given in Section 6. Also in this section, the collocation method for the weakly singular Hammerstein equation is discussed, extending the results of Section 3. The method of the singularity preserving Galerkin method is discussed in Section 7. **Part 1**

2 The degenerate kernel method

In this section, we are concerned with the problem of finding an approximate solution of Eqn. (1) by the degenerate kernel method. The existence of a unique solution is guaranteed by an application of the Banach contraction principle under the following assumptions:

1. $k \in C([0,1] \times [0,1])$
2. $\psi \in C([0,1] \times (-\infty, \infty))$ and

$$\left\{ \int_0^1 |\psi(s, x(s))|^2 ds \right\}^{1/2} \leq A\|x\|_2,$$

   where $x \in L_2[0,1]$ and A is a constant independent of x,

3. ψ satisfies the Lipschitz condition $|\psi(t, x_1) - \psi(t, x_2)| \leq B|x_1 - x_2|$ where $x_1, x_2 \in (-\infty, \infty)$ and B is a constant independent of x_1 and x_2,
4. k is bounded by $|k(t,s)| < C$ with $BC < 1$

As described in [5], the degenerate kernel method presented in this section can be applied to Hammerstein equations with multiple solutions. Hence, the conditions above, which guarantee the global uniqueness of a solution, can be relaxed for an application of the method. Suppose that k_n is an approximation of the kernel k in Eqn. (1) that has the following form,

$$k_n(t,s) = \sum_{i=1}^{n} B_i(t) C_i(s), \tag{2}$$

where $\{B_i\}_{i=1}^n$ is a linearly independent set of functions in $C[0,1]$ and $\{C_i\}_{i=1}^n$ is a set of functions from $C[0,1]$. We assume that

$$\left\{ \int_0^1 \int_0^1 |k_n(t,s) - k(t,s)|^2 dt ds \right\}^{1/2} \longrightarrow 0 \qquad \text{as } n \to \infty. \tag{3}$$

In the degenerate kernel method, an approximate solution x_n is obtained by solving the following equation,

$$x_n(t) - \int_0^1 k_n(t,s) \psi(s, x_n(s))\, ds = f(t), \qquad t \in [0,1]. \tag{4}$$

Replacing k_n by the expression of the right hand side in Eqn. (2), we obtain

$$x_n(t) - \sum_{i=1}^{n} B_i(t) \int_0^1 C_i(s)\psi(s, x_n(s))ds = f(t), \qquad t \in [0,1]. \tag{5}$$

Let

$$c_i = \int_0^1 C_i(s)\psi(s, x_n(s))ds.$$

Then from Eqn. (5),

$$x_n(t) = f(t) + \sum_{i=1}^{n} c_i B_i(t), \tag{6}$$

where c_i's are constants that can be determined by solving the following set of nonlinear equations.

$$c_j = \int_0^1 C_j(t)\psi(t, f(t) + \sum_{i=1}^{n} c_i B_i(t))dt, \qquad \text{for } j = 1, 2, \ldots, n. \tag{7}$$

The accuracy of approximation of the degenerate kernel method depends upon the degree of approximation that k_n makes for k in Eqn. (3). The following theorem from [5] summarizes this point.

Theorem 2.1 *Let $k_n \in C([0,1] \times [0,1])$ satisfy Eqn. (2). Then equation (4) has a unique solution $x_n \in L_2[0,1]$ for all sufficiently large n. Moreover,*

$$\|x - x_n\|_2 \leq \frac{A\|x_n\|_2}{1 - BC}\|k - k_n\|_2.$$

Examples: (A) Consider

$$x(t) - \int_0^1 tsx^2(s)ds = \frac{3}{4}t, \qquad t \in [0,1].$$

This equation possesses multiple solutions. They are $x_1(t) = t$ and $x_2(t) = 3t$. The kernel is already degenerate. Thus we take $B_1(t) = t$ and $C_1(s) = s$. The solutions to $c^2 - \frac{5}{2} + \frac{9}{16} = 0$ are used in $x_n(t) = \frac{3}{4}t + ct$ to obtain the exact solutions.

(B) Consider

$$x(t) - \int_0^1 e^{ts}e^{-x^2(s)}ds = \sqrt{t} - \frac{e^{t-1} - 1}{t - 1}, \qquad t \in [0,1].$$

The kernel can be approximated by several different methods. For instance,

$$e^{ts} \sim 1 + ts + \frac{t^2s^2}{2!} + \cdots + \frac{t^n s^n}{n!}$$

corresponding to the Taylor approximation, or

$$e^{ts} \sim \sum_{i=0}^{n} \sum_{j=0}^{n} e^{t_i s_j} B_i(t) C_j(s)$$

where B_i and C_j linear splines with respective knots $0 \le t_1 < t_2 < \cdots < t_n \le 1$ and $0 \le s_1 < s_2 < \cdots < s_n \le 1$. We obtain the following results for each of the two approximation schemes described above. The errors are computed in $\|\cdot\|_2$ norm. **Table 1.**

	Errors	
n	Taylor	spline
2	7.218e-2	6.203e-2
3	9.847e-3	1.553e-2
4	2.013e-3	3.768e-3

3 The projection methods

In this section, the collocation and Galerkin methods for Hammerstein equations will be described as two special cases of the projection method. First, we discuss the Galerkin method.

3.1 The Galerkin Method

Let n be a positive integer and $\{X_n\}$ be a sequence of finite dimensional subspaces of $C[0,1]$ such that for any $x \in C[0,1]$ there exists a sequence $\{x_n\}$, $x_n \in X_n$, for which

$$\|x_n - x\|_\infty \to 0 \;\text{ as } n \to \infty. \tag{8}$$

Let $P_n^G \colon L_2[0,1] \to X_n$ be an orthogonal projection for each n. We assume that the projection P_n^G when restricted to $C[0,1]$ is uniformly bounded, i.e.

$$P := \sup_n \|P_n^G|_{C[0,1]}\|_\infty < \infty. \tag{9}$$

Then from Eqns. (8) and (9), it follows that for each $x \in C[0,1]$,

$$\|P_n^G x - x\|_\infty \to 0, \;\text{ as }\; n \to \infty. \tag{10}$$

If we put

$$(K\Psi)(x)(t) \equiv \int_0^1 k(t,s)\psi(s, x(s))ds,$$

then Eqn. (1) takes the following operator form

$$x - K\Psi x = f. \tag{11}$$

As indicated in the previous section, Eqn. (11) may admit multiple solutions. Hence it is assumed for the remainder of this paper that we are treating a solution x_0 of Eqn. (11) that is isolated.

Let $\{\varphi_{nj}\}_{j=1}^{n}$ be a set of linearly independent functions that spans X_n. The Galerkin method is to find

$$x_n = \sum_{j=1}^{n} b_{nj}\varphi_{nj}$$

that satisfies

$$x_n - P_n^G K\Psi x_n = P_n^G f. \tag{12}$$

Similarly one is required to find b_{nj}'s that satisfy the system of nonlinear equations described by

$$\begin{aligned} \textstyle\sum_{j=1}^{n} b_{nj} < \varphi_{nj}, \varphi_{ni} > \quad & - < \textstyle\int_0^1 k(\cdot,s)\psi(s, \sum_{j=1}^{n} b_{nj}\varphi_{nj}(s))ds, \varphi_{ni} > \\ & =< f, \varphi_{ni} >, \qquad 1 \le i \le n, \end{aligned} \tag{13}$$

where $< .,. >$ denotes the standard inner product in L_2.

Now we let

$$Tx \equiv f + K\Psi x \tag{14}$$

and

$$T_n^G x_n \equiv P_n^G f + P_n^G K\Psi x_n \tag{15}$$

so that Eqns. (11) and (12) can be written respectively as $x = Tx$ and $x_n = T_n^G x_n$. A proof of the following theorem can be made by directly applying Theorem 2 of Vainikko [10]. The paper of Atkinson and Potra [11] is also useful in this context.

Theorem 3.1 *Let $x_0 \in C[0,1]$ be an isolated solution of Eqn. (11). Assume that 1 is not an eigenvalue of the linear operator $(K\Psi)'(x_0)$, where $(K\Psi)'(x_0)$ denotes the Fréchet derivative of $K\Psi$ at x_0. Then the Galerkin approximation Eqn. (12) has a unique solution $x_n \in B(x_0,\delta)$ for some $\delta > 0$ and for sufficiently large n. Moreover, there exists a constant $0 < q < 1$, independent of n, such that*

$$\frac{\alpha_n}{1+q} \le \|x_n - x_0\|_\infty \le \frac{\alpha_n}{1-q}, \tag{16}$$

where $\alpha_n \equiv \|(I - T_n^{G'}(x_0))^{-1}(T_n^G(x_0) - T(x_0))\|_\infty$. Finally,

$$E_n(x_0) \le \|x_n - x_0\|_\infty \le C E_n(x_0), \tag{17}$$

where C is a constant independent of n and $E_n(x_0) = \inf_{u \in S_n^r} \|x_0 - u\|_\infty$.

We denote by $W_p^m[0,1]$, $1 \le p \le \infty$, the Sobolev space of functions g whose m-th generalized derivative $g^{(m)}$ belongs to $L_p[0,1]$. The space $W_p^m[0,1]$ is equipped with the norm

$$\|g\|_{W_p^m} \equiv \sum_{k=0}^{m} \|g^{(k)}\|_p.$$

We now specify the finite dimensional subspace X_n. For any positive integer n, let

$$\Pi_n : 0 = t_0 < t_1 < ... < t_{n-1} < t_n = 1 \tag{18}$$

be a partition of $[0,1]$. Let r and ν be nonnegative integers satisfying $0 \le \nu < r$. Let $S_r^\nu(\Pi_n)$ denote the space of splines of order r, continuity ν, with knots at Π_n, that is

$$S_r^\nu(\Pi_n) = \{x \in C^\nu[0,1] : x|_{[t_i,t_{i+1}]} \in \mathcal{P}_{r-1}, \text{ for each } i = 0,1,\ldots,n-1\}$$

where $\mathcal{P}_{r-1}$ denotes the space of polynomials of degree $\le r-1$. We assume that the sequence of partitions Π_n of $[0,1]$ satisfies the condition that there exists a constant $C > 0$, independent of n, with the property:

$$\frac{\max_{1\le i\le n}(t_i - t_{i-1})}{\min_{1\le i\le n}(t_i - t_{i-1})} \le C, \text{ for all } n. \tag{19}$$

It is known from de Boor [12] and Douglas, Dupont and Wahlbin [13] that condition (19) implies that the Galerkin projections P_n are uniformly bounded. In addition, it is also well known from Demko [14] and De Vore [15] that if $0 \le \nu < r$, $1 \le p \le \infty$, $m \ge 0$ and $x \in W_p^m$, then for each $n \ge 1$, there exists $u_n \in S_r^\nu(\Pi_n)$ such that

$$\|x - u_n\|_p \le Ch^\mu \|x\|_{W_p^\mu}, \tag{20}$$

where $\mu = \min\{m, r\}$ and $h = \max_{1\le i\le n}(t_i - t_{i-1})$. Using Theorem 3.1 and the inequalities (17) and (20), we obtain the following theorem.

Theorem 3.2 *Let x_0 be an isolated solution of Eqn. (11) and let x_n be the solution of Eqn. (12) in a neighborhood of x_0. Assume that* 1 *is not an eigenvalue of $(K\Psi)'(x_0)$. If $x_0 \in W_\infty^l$ $(0 \le l \le r)$, then*

$$\|x_0 - x_n\|_\infty = O(h^\mu),$$

where $\mu = \min\{l, r\}$. If $x_0 \in W_p^l$ $(0 < l \le r,\ 1 \le p < \infty)$, then

$$\|x_0 - x_n\|_\infty = O(h^\nu),$$

where $\nu = \min\{l-1, r\}$.

We remark that the Galerkin method for Urysohn equations was obtained by Atkinson and Potra [11]. Hence, Theorem 3.2 may be derived by specializing their result to Hammerstein equations.

3.2 The Collocation Method Let Π_n denote the partition of $[0,1]$ defined in Eqn. (18). For each positive integer r, define a sequence of points $\{\xi_i\}_{i=0}^r$ such that $0 \leq \xi_0 < \xi_1 < \cdots < \xi_r \leq 1$. Also

$$t_{ij} \equiv t_i + \xi_j(t_{i+1} - t_i), \qquad i = 0, 1, \ldots, n-1; \quad j = 0, 1, \ldots r, \tag{21}$$

so that

$$t_i \leq t_{i0} < t_{i1} < \cdots < t_{ir} \leq t_{i+1}, \qquad i = 0, 1, \ldots, n-1.$$

In the collocation method, the approximate solution is constructed in $S_r^\nu(\Pi_n)$ by the following strategy. For each $i = 0, 1, \ldots, n-1$, let l_{ij} denote the Lagrange fundamental polynomial for the knots $\{t_{ij}\}_{j=0}^r$ defined by

$$l_{ij}(s) = \prod_{l=1, l\neq j}^{r} \frac{s - t_{il}}{t_{ij} - t_{il}}, \qquad t_i \leq s \leq t_{i+1}.$$

We seek the approximate solution $x_n \in S_r^\nu$ in the form $x_n(t) = \sum_{j=0}^r a_{ij} l_{ij}(t)$, for $t_i \leq t \leq t_{i+1}$, $i = 0, 1, \ldots, n-1$ by solving the following set of nonlinear equations for a_{ij},

$$a_{ij} - \sum_{p=0}^{n-1} \int_{t_p}^{t_{p+1}} k(t_{ij}, s)\psi(s, \sum_{k=0}^{r} a_{pk} l_{pk}(s)) ds = f(t_{ij}), \tag{22}$$

for $i = 0, 1, \ldots, n-1$ and $j = 0, 1, \ldots, r$. The interpolation projector $P_n^C : C[0,1] \to S_r^\nu$ is defined by

$$P_n^C \varphi(t) = \sum_{k=0}^{r} \varphi(t_{ik}) l_{ik}(t), \tag{23}$$

for $\varphi \in C[0,1]$ and for $t_i \leq t \leq t_{i+1}$, $i = 0, 1, \ldots, n-1$. Using the notations introduced above, Eqn. (22) can be described symbolically as

$$x_n - P_n^C K \Psi x_n = P_n^C f. \tag{24}$$

Under the assumption that $h \equiv \max_{0 \leq i \leq n-1}(t_{i+1} - t_i) \to 0$ as $n \to \infty$, we have

$$\|P_n^C \varphi - \varphi\|_\infty \to 0, \qquad \text{as } n \to \infty. \tag{25}$$

The following theorem which proves the existence of the collocation solution is analogous to Theorem 3.1. We list it here for purpose of completeness. Define $T_n^C x_n \equiv P_n^C K \Psi x_n + P_n^C f$.

Theorem 3.3 *Let $x_0 \in C[0,1]$ be an isolated solution of Eqn. (11). Assume that 1 is not an eigenvalue of the linear operator $(K\Psi)'(x_0)$, where $(K\Psi)'(x_0)$ denotes the Fréchet derivative of $K\Psi$ at x_0. Then the collocation approximation Eqn. (24) has a unique solution $x_n \in B(x_0, \delta)$ for some $\delta > 0$ and for sufficiently large n. Moreover, there exists a constant $0 < q < 1$, independent of n, such that*

$$\frac{\alpha_n}{1+q} \le \|x_n - x_0\|_\infty \le \frac{\alpha_n}{1-q}, \tag{3.19}$$

where $\alpha_n \equiv \|(I - T_n^{C'}(x_0))^{-1}(T_n^C(x_0) - T(x_0))\|_\infty$. Finally,

$$E_n(x_0) \le \|x_n - x_0\|_\infty \le C E_n(x_0), \tag{3.20}$$

where C is a constant independent of n and $E_n(x_0) = \inf_{u \in S_r^\nu(\Pi_n)} \|x_0 - u\|_\infty$.

To obtain the convergence and the rate of convergence of the collocation approximation, we argue from Eqn. (26) as follows;
Proof:

$$\begin{aligned}
\|x_n - x_0\|_\infty &\le \tfrac{\alpha_n}{1-q} \\
&= \tfrac{\|(I-T_n^{C'}(x_0))^{-1}(T_n^C(x_0)-T(x_0))\|_\infty}{1-q} \\
&\le \tfrac{\|(I-T_n^{C'}(x_0))^{-1}\|_\infty \|(T_n^C(x_0)-T(x_0))\|_\infty}{1-q} \\
&= C\|P_n^C K\Psi(x_0) - K\Psi(x_0) + P_n^C f - f\|_\infty \\
&= C\|P_n^C(x_0) - x_0\|_\infty,
\end{aligned}$$

where C is a constant independent of n. From this and Eqn. (25) along with (20), if $x_0 \in W_\infty^r$, then

$$\|x_0 - x_n\|_\infty = O(h^r). \tag{3.21}$$

4 Superconvergence of the iterates

In this section, we review some results concerning the superconvergence of the iterated Galerkin and the iterated collocation methods for Hammerstein equations. The results presented here are taken from the recent papers of Kaneko and Xu [16], and Kaneko, Noren and Padilla [17]. The superconvergence phenomena of the iterates for the Fredholm equations was originally studied by Sloan [18] and it was extended by him and by his collaborators [19-26]. Some of their results are generalized in this section to hold for the Hammerstein equation. The discussion on the iterated Galerkin method is given below. A discussion that is pertinent to the iterated collocation method for Hammerstein equations is quite similar. Therefore only the points that distinguish the iterated collocation method from that of the Galerkin method will be given here.

Throughout this section, in addition to the four assumptions that are described at the beginning of Section 2, we also assume the following two additional conditions; 5. the partial derivative $\psi^{(0,1)}$ of ψ with respect to the second variable exists and is Lipschitz continuous, i.e., there exists a constant $C_2 > 0$ such that

$$|\psi^{(0,1)}(t, x_1) - \psi^{(0,1)}(t, x_2)| \leq C_2 |x_1 - x_2|, \text{ for all } x_1, x_2 \in (-\infty, \infty);$$

6. for $x \in C[0, 1]$, $\psi(., x(.))$, $\psi^{(0,1)}(., x(.)) \in C[0, 1]$.

We assume that x_n is the unique solution of Eqn. (12) in the sphere $B(x_0, \delta)$ for some $\delta > 0$. Define

$$x_n^I = f + K\Psi x_n. \tag{29}$$

Applying P_n^G to the both sides of (29), we obtain

$$P_n^G x_n^I = P_n^G f + P_n^G K\Psi x_n. \tag{30}$$

From Eqns. (30) and (12), we see that

$$P_n^G x_n^I = x_n. \tag{31}$$

Hence the function x_n^I satisfies the following new Hammerstein equation

$$x_n^I = f + K\Psi P_n^G x_n^I. \tag{32}$$

By letting $S_n^G \equiv f + K\Psi P_n^G$, we may rewrite Eqn. (32) as $x_n^I = S_n^G x_n^I$. The following two lemmas are instrumental to Theorem 4.3 below which proves the superconvergence of the iterated Galerkin method.

Lemma 4.1 *Let $x_0 \in C[0, 1]$ be an isolated solution of Eqn. (11). Assume that 1 is not an eigenvalue of $(K\Psi)'(x_0)$. Then for sufficiently large n, the operators $I - (S_n^G)'(x_0)$ are invertible and there exists a constant $L > 0$ such that*

$$\|(I - (S_n^G)'(x_0))^{-1}\|_\infty \leq L, \text{ for sufficiently large } n.$$

Lemma 4.2 *Let $x_0 \in C[0, 1]$ be an isolated solution of Eqn. (11) and x_n be the unique solution of Eqn. (12) in the sphere $B(x_0, \delta_1)$. Assume that 1 is not an eigenvalue of $(K\Psi)'(x_0)$. Then for sufficiently large n, x_n^I defined by the iterated scheme Eqn. (29) is the unique solution of Eqn. (32) in* *the sphere* $B(x_0, \delta)$*, for some* $\delta > 0$*. Moreover, there exists a constant* $0 < q < 1$*, independent of* n*, such that*

$$\frac{\beta_n}{1+q} \le \|x_n^I - x_0\|_\infty \le \frac{\beta_n}{1-q},$$

where $\beta_n = \|(I - (S_n^G)'(x_0))^{-1}[S_n^G(x_0) - T(x_0)]\|_\infty$*. Finally,*

$$\|x_n^I - x_0\|_\infty \le CE_n(x_0).$$

First, we apply the mean-value theorem to $\psi(s, y)$ to conclude

$$\psi(s, y) = \psi(s, y_0) + \psi^{(0,1)}(s, y_0 + \theta(y - y_0))(y - y_0), \tag{33}$$

where $\theta := \theta(s, y_0, y)$ with $0 < \theta < 1$. The boundedness of θ is essential for the proof of the next theorem, although it may depend on s, y_0, y. Let

$$g(t, s, y_0, y, \theta) = k(t, s)\psi^{(0,1)}(s, y_0 + \theta(y - y_0)),$$

$$(G_n x)(t) = \int_0^1 g(t, s, P_n^G x_0(s), P_n^G x_n'(s), \theta)x(s)ds,$$

and $(Gx)(t) = \int_0^1 g_t(s)x(s)ds$, where $g_t(s) = k(t, s)\psi^{(0,1)}(s, x_0(s))$.

Theorem 4.3 *Let* $x_0 \in C[0, 1]$ *be an isolated solution of Eqn. (11) and* x_n *be the unique solution of Eqn. (12) in the sphere* $B(x_0, \delta)$ *for some* $\delta > 0$*. Let* x_n^I *be defined by the iterated scheme Eqn. (29). Assume that* 1 *is not an eigenvalue of* $(K\Psi)'(x_0)$*. Then, for all* $1 \le p \le \infty$*,*

$$\begin{aligned}\|x_0 - x_n^I\|_\infty \le \; & C\{\|x_0 - P_n^G x_0\|_\infty^2 \\ & + \sup_{0 \le t \le 1} \inf_{u \in X_n} \|k(t, \cdot)\psi^{(0,1)}(\cdot, x_0(\cdot)) - u\|_q \|x_0 - P_n^G x_0\|_p\},\end{aligned}$$

where $1/p + 1/q = 1$ *and* C *is a constant independent of* n*.*

Proof: From Eqns. (11) and (32) we have

$$\begin{aligned} x_0 - x_n &= K(\Psi x_0 - \Psi P_n^G x_n^I) \\ &= K(\Psi x_0 - \Psi P_n^G x_0) + K(\Psi P_n^G x_0 - \Psi P_n^G x_n^I) \\ &= K(\Psi x_0 - \Psi P_n^G x_0)(G_n P_n^G (x_0 - x_n^I))(t). \end{aligned} \tag{34}$$

By using assumption (5) and the fact $0 < \theta < 1$, we have, for all $x \in C[0, 1]$,

$$\begin{aligned}\|(G_n x) - (Gx)\|_\infty \; & \le \sup_{0 \le t \le 1} \int_0^1 |k(t, s)|ds \|x\|_\infty \times \\ & (\|P_n^G x_0 - x_0\|_\infty + \|P_n^G\|_\infty \|x_n^I - x_0\|_\infty).\end{aligned}$$

Consequently, by Eqn. (10) and Lemma 4.2,

$$\|G_n - G\|_\infty \le M(\|P_n^G x_0 - x_0\|_\infty + P\|x_n^I - x_0\|_\infty) \to 0 \ \text{ as } \ n \to \infty.$$

for some constant $M > 0$. Hence $G_n \to G$ in the norm of $C[0,1]$ as $n \to \infty$. Moreover, for each $x \in C[0,1]$,

$$\begin{aligned} \sup_{0\le t\le 1} |(GP_n^G x)(t) - (Gx)(t)| &= \sup_{0\le t\le 1} |\textstyle\int_0^1 g_t(s)[P_n^G x(s) - x(s)]ds| \\ &\le MM_1\|P_n x - x\|_\infty, \end{aligned}$$

where

$$M_1 = \sup_{0\le t\le 1} |\psi^{(0,1)}(t, x_0(t))| < +\infty.$$

It follows that $GP_n^G \to G$ pointwise in $C[0,1]$ as $n \to \infty$. Again since P_n^G is uniformly bounded, we have for each $x \in C[0,1]$,

$$\|G_n P_n^G x - Gx\|_\infty \le \|G_n - G\|_\infty \|P_n^G\|_\infty \|x\|_\infty + \|GP_n^G x - Gx\|_\infty.$$

Thus, $G_n P_n^G \to G$ pointwise in $C[0,1]$ as $n \to \infty$. By Assumptions 5 and 6, we see that there exists a constant $C > 0$ such that for all n

$$\begin{aligned} &|\psi^{(0,1)}(s, P_n^G x_0(s) + \theta(P_n^G x_n^I(s) - P_n^G x_0(s)))| \\ &\qquad \le C_2\|P_n^G x_0 - x_0\|_\infty + \theta C_2 P\|x_n^I - x_0\|_\infty + M_1 \le C. \end{aligned}$$

It can be shown that $\{G_n P_n^G\}$ is collectively compact. Since $G = (K\Psi)'(x_0)$ is compact and $(I-G)^{-1}$ exists, it follows from the theory of collectively compact operators [1] that $(I - G_n P_n^G)^{-1}$ exists and is uniformly bounded for sufficiently large n. By Eqn. (34), we have the following estimate

$$\sup_{0\le t\le 1} |(x_0 - x_n^I)(t)| \le C \sup_{0\le t\le 1} |K(\Psi x_0 - \Psi P_n^G x_0)(t)|. \tag{35}$$

Next, we estimate the function $d(t) \equiv |K(\Psi x_0 - \Psi P_n^G x_0)(t)|$. Using Eqn. (33) with $y = P_n^G x_0$ and $y_0 = x_0$, we obtain, for $0 < \theta < 1$,

$$d(t) = \left| \int_0^1 g(t, s, x_0(s), P_n^G x_0(s), \theta)(x_0(s) - P_n^G x_0(s))ds \right|.$$

Note that $\int_0^1 u(s)[x_0(s) - P_n^G x_0(s)]ds = 0$, for all $u \in X_n$. Thus, for all $u \in X_n$,

$$\begin{aligned} d(t) &= \left| \int_0^1 [g(t, s, x_0(s), P_n^G x_0(s), \theta) - u(s)](x_0(s) - P_n^G x_0(s))ds \right| \\ &\le \int_0^1 |g(t, s, x_0(s), P_n^G x_0(s), \theta) - g_t(s)|ds \|x_0 - P_n^G x_0\|_\infty \\ &\qquad + \left| \int_0^1 [g_t(s) - u(s)](x_0(s) - P_n^G x_0(s))ds \right|. \end{aligned}$$

Now, by assumption (5), we have

$$\begin{aligned}\int_0^1 |g(t,s,x_0,P_n^G x_0(s),\theta)-g_t(s)|ds &\le C_1\theta\int_0^1 |k(t,s)|ds\|x_0-P_n^G x_0\|_\infty \\ &\le C_1 M\|x_0-P_n^G x_0\|_\infty.\end{aligned}$$

Moreover, for $1/p+1/q=1$,

$$\left|\int_0^1 [g_t(s)-u(s)][x_0(s)-P_n^G x_0(s)]ds\right| \le \|g_t-u\|_q\|x_0-P_n^G x_0\|_p.$$

Therefore,

$$d(t)\le C_1 M\|x_0-P_n^G x_0\|_\infty^2+\|g_t-u\|_q\|x_0-P_n^G x_0\|_p, \quad \text{for all} \quad u\in X_n.$$

This proves the theorem. □

The next theorem is concerned with the case $X_n = S_r^\nu(\Pi_n)$ where Π_n satisfies Eqn. (19).

Theorem 4.4 *Let $x_0 \in W_p^l$ $(0<l\le r)$ be an isolated solution of Eqn. (11), x_n be the unique solution of Eqn. (12) in $B(x_0,\delta)$, for some $\delta>0$. Let x_n^I be defined by the iterated scheme Eqn. (29). Assume that 1 is not an eigenvalue of $(K\Psi)'(x_0)$. Assume also that for all $t\in[0,1]$, $k_t(.)\psi^{(0,1)}(.,x_0(.))\in W_q^m(0\le m\le r)$. Then*

$$\|x_0-x_n^I\|_\infty = O(h^{\mu+\min\{\mu,\nu\}}),$$

where $\mu=\min\{l,r\}$ and $\nu=\min\{m,r\}$.

If, in Eqn. (29), x_n denotes the collocation solution, then the corresponding x_n^I satisfies

$$x_n^I = f + K\Psi P_n^C x_n^I. \tag{36}$$

The following theorem for the superconvergence of the iterated collocation method is proved in [17].

Theorem 4.5 *Let $x_0\in C[0,1]$ be an isolated solution of equation Eqn. (11) and x_n be the unique solution of Eqn. (24) in the sphere $B(x_0,\delta_1)$. Let x_n^I be defined by the iterated scheme Eqn. (29). Assume that 1 is not an eigenvalue of $(K\Psi)'(x_0)$. Assume that $x_0\in W_1^l$ $(0<l\le 2r)$ and $g_t\in W_1^m$ $(0<m\le r)$ with $\|g_t\|_{W_1^m}$ bounded independently of t. Then*

$$\|x_0-x_n^I\|_\infty = O(h^\gamma), \qquad \textit{where } \gamma = min\{l, r+m\}.$$

Proof: The first part of the proof for this theorem given in [17] follows the same way as that of Theorem 4.3 up to equation Eqn. (35). Thereafter, we invoke to the following four known inequalities. Let $\psi_n \in S_l^0(\Pi_n)$ be such that

$$\sum_{i=1}^{n} \|(x_0 - \psi_n)^{(j)}\|_{W_1^m(I_i)} \leq ch^{l-j} \|x_0\|_{W_1^l}, \qquad 0 \leq j \leq l, \tag{37}$$

$$\max_{1 \leq i \leq n} \|\psi_n^{(j)}\|_{W_\infty^m(I_i)} \leq c\|x_0\|_{W_1^l}, \qquad j \geq 0. \tag{38}$$

Also for each $t \in [0,1]$, there exists $\varphi_{n,t} \in S_m^0(\Pi_n)$ such that

$$\sum_{i=1}^{n} \|(g_t - \varphi_{n,t})^{(j)}\|_{W_1^m(I_i)} \leq ch^{m-j} K_m, \qquad 0 \leq j \leq m, \tag{39}$$

$$\max_{1 \leq i \leq n} \|\varphi_{n,t}^{(j)}\|_{W_\infty^m(I_i)} \leq cK_m, \qquad j \geq 0, \tag{40}$$

where $K_m = \sup_{0 \leq t \leq 1} \|k_t\|_{W_1^m} < \infty$. Now for $t \in [0,1]$ we have

$$\begin{aligned} K(\Psi x_0 - \Psi P_n x_0)(t) &= (g_t - \varphi_{n,t}, x_0 - P_n x_0) + (\varphi_{n,t}, (I - P_n)(x_0 - \psi_n)) \\ &\quad + (\varphi_{n,t}, (I - P_n)\psi_n). \end{aligned} \tag{41}$$

Using Eqns. (37)-(40) along with the arguments from [19] (p.362) we can show that each of the three terms is bounded by ch^γ uniformly in t. This completes the proof. □

Examples supporting the results given in Theorems 4.3 and 4.5 can be found in [16],[17].

PART II - Weakly Singular Hammerstein Equations

5 Regularities of the solutions

In this section we are concerned with the regularity properties of the solution to Eqn. (1) when the kernel $k(t,s)$ is weakly singular. In particular we assume that

$$k(s,t) = g_\alpha(|s-t|)m(s,t), \tag{5.1}$$

where $m \in C([0,1] \times [0,1])$. (further hypotheses on m appear later), and

$$g_\alpha(|s-t|) = \begin{cases} |s-t|^{\alpha-1}, & 0 < \alpha < 1, \\ \log|s-t|, & \alpha = 1. \end{cases} \tag{5.2}$$

We further assume that

$$\psi \in C([0,1] \times (-\infty, \infty)) \tag{5.3}$$

$$|\psi(t, y_1) - \psi(t, y_2)| \leq A|y_1 - y_2|. \tag{5.4}$$

There is a large literature, e.g. [41][42], on existence of solution of Hammerstein equations. The following theorem is typical of such results.

Theorem 5.1 *Let $f \in C[0,1]$ and assume*

$$A \cdot \sup_{0\le\alpha\le 1} \int_0^1 |g_\alpha(s-t)m(s,t)|dt < 1.$$

Then the Hammerstein equation with weakly singular kernel has a unique solution in $C[0,1]$.

Define for $0 < \alpha \le 1$ and nonnegative integer m the set $C^{(m,\alpha)}[0,1]$ of all functions $x \in C^m[0,1]$ such that there exists constants $A > 0$ and $B > 1$ with

$$|x^{(m)}(s) - x^{(m)}(t)| \le A \cdot \begin{cases} |s-t|^\alpha & ,0<\alpha<1, \\ |s-t|\log(\frac{B}{|s-t|}) & ,\alpha = 1. \end{cases}$$

Then x is called α-Hölder continuous. A nonstandard way of defining Hölder continuity for $\alpha = 1$ should be noted here.

The proof of Theorem 5.3 below uses the following lemma in [14].

Lemma 5.2 *Let $m \in C^1([0,1]\times[0,1])$. Then*

(i) If $x \in C[0,1]$, $Kx \in C^{(0,\alpha)}[0,1]$;

(ii) if $x \in C^{(0,\mu)}$, $0 \le \mu \le 1-\alpha < 1$, $|m_\alpha(s)-m_\alpha(t)| \le |s-t|^{\alpha+\mu}$, $s,t \in [0,1]$ where $m_\alpha \equiv \int_0^1 g_\alpha(|s-t|)m(s,t)dt$, then $Kx \in C^{(0,\alpha+\mu)}[0,1]$;

(iii) if $x \in C^{(0,\mu)}[0,1]$,$0 \le 1-\alpha < \mu \le 1$, $\lim_{r\to s} x(s)\frac{(m_\alpha(s)-m_\alpha(r))}{s-r}$,$r \in [0,1]$, exists for all $s \in [0,1]$, and continuous in s, then $Kx \in C^1[0,1]$ and $\frac{d}{ds}Kx(s) = \int_0^1 \frac{\partial}{\partial s}\{g_\alpha(|s-t|)m(s,t)\}[x(t)-x(s)]dt + x(s)\frac{d}{ds}m_\alpha(s)$.

The next theorem generalizes the result in [26].

Theorem 5.3 *Let n be a nonnegative integer, $m \in C^{n+1}([0,1]\times[0,1])$, $f \in C^{(0,\alpha)}[0,1] \cap C^n(0,1)$ and $f_i(s) \equiv s^i(1-s)^i f^{(i)}(s)$,$i = 1,\dots,n$ be α-Hölder continuous on $[0,1]$. For $n = 0,1$, we assume that $\psi \in C^{(0,1)}([0,1]\times(-\infty,\infty))$ and for $n \ge 2$, we assume that $\psi \in C^{(n-1)}([0,1]\times(-\infty,\infty))$. If x is any solution of (1.4), then it belongs to $C^n(0,1)\cap C^{(0,\alpha)}[0,1]$. Moreover, $x_i(s) \equiv s^i(1-s)^i\psi^{(i)}(s)$ belongs to $C^{(0,\infty)}[0,1]$ for $i = 1,2,\dots n$.*

The proof requires the following lemma.

Lemma 5.4 *(i)* $(s-t)\frac{\partial}{\partial s}g_\alpha(|s-t|) = \begin{cases} (\alpha-1)g_\alpha(|s-t|), & 0<\alpha<1, \\ 1, & \alpha = 1; \end{cases}$

(ii) $\frac{\partial}{\partial s}\int_0^t g_\alpha(|s-y|)dy = g_\alpha(s) - g_\alpha(|s-t|)$;

(iii)

$$\frac{d}{ds}m_\alpha(s) = m(s,0)g_\alpha(s) - m(s,1)g_\alpha(1-s) + \int_0^1 \frac{\partial k(s,t)}{\partial s}g_\alpha(|s-t|)dt + \int_0^1 \frac{\partial m(s,t)}{\partial t}g_\alpha(|s-t|)dt.$$

Now we are ready to present a proof of Theorem 5.3.
Proof: For $n = 0$, because of Lemma 5.2 (i), the result is true. For $n = 1$, multiplication by $h(t) \equiv t(1-t)$ gives

$$h(t)x(t) - h(t)K\Psi(x)(t) = h(t)f(t), \qquad 0 \leq t \leq 1. \tag{5.5}$$

By assumption $hf \in C^{(1,\alpha)}[0,1]$. Note that $h(t)\psi(x)(t) = Kh\Psi(x)(t) + \int_0^1 g_\alpha(|s-t|)m(t,s)\psi(s,x(s))(t-s)(1-t-s)ds$. If we define this last term as $\hat{K}\Psi(x)(t)$, then (5.5) becomes

$$h(t)x(t) - Kh\Psi(x)(t) = h(t)f(t) + \hat{K}\Psi(x)(t), \qquad 0 \leq t \leq 1. \tag{5.6}$$

The multiplication by $(t-s)$ in the kernel $\hat{K}\Psi$ makes $\hat{K}\Psi(x)$ smoother than $K\Psi(x)$. For more information on this point see [24]. Hence $hf + \hat{K}\Psi(x) \in C^1[0,1]$. It follows from Lemma 5.2 that $Kh\Psi(x) \in C^{(0,\alpha)}[0,1]$ and by (5.6) that $hx \in C^{(0,\alpha)}$. In the remainder of the section, M denotes a constant whose value may change each time it appears. Now for $t_1, t_2 \in [0,1]$,

$$\begin{aligned}
&|h(t_1)\psi(t_1,x(t_1)) - h(t_2)\psi(t_2,x(t_2))| \\
&\leq |h(t_1)||\psi(t_1,x(t_1)) - \psi(t_2,x(t_2))| + |h(t_1)||\psi(t_1,x(t_2)) - \psi(t_2,x(t_2))| \\
&\qquad + |h(t_1) - h(t_2)||\psi(t_2,x(t_2))| \\
&\leq M|h(t_1)(x(t_1) - x(t_2))| + M|h(t_2)||t_1 - t_2| \\
&\qquad + |\psi(t_2,x(t_2))||t_1 - t_2| \\
&\leq M|h(t_1)x(t_1) - h(t_2)x(t_2)| + M|x(t_2)||h(t_2) - h(t_1)| \\
&\qquad + M|t_1 - t_2|.
\end{aligned} \tag{5.7}$$

Since $hx \in C^{(0,\alpha)}[0,1]$, we get $h\psi(x) \in C^{(0,\alpha)}[0,1]$. If $\alpha > \frac{1}{2}$, let $\mu = \alpha$. Then $0 \leq 1-\alpha < \mu \leq 1$ and $h\psi(x) \in C^{(0,\alpha)}[0,1]$. If $0 < \alpha \leq \frac{1}{2}$, then Lemma 5.4 (ii) yields $h\psi(x)\frac{dm_\alpha(s)}{ds} \in C[0,1]$, and then Lemma 5.2 (ii) implies $Kh\Psi(x) \in C^{(0,2\alpha)}[0,1]$. By (5.6), $hx \in C^{(0,2\alpha)}[0,1]$. The argument may be repeated to ensure the existence of μ with $0 \leq 1-\alpha < \mu \leq 1$ such that $h\Psi(x) \in C^{(0,\mu)}[0,1]$. By Lemma 5.2 (iii), it follows that $Kh\Psi(x) \in C^1[0,1]$. By application of (5.6), we get $hx \in C^1[0,1]$. Finally, $x \in C^1(0,1) \cap C[0,1]$. To complete the proof, integrate by parts in (1.1) to obtain

$$\begin{aligned}
&x(t) + \int_0^1 G_\alpha(|t-s|)m(t,s)\left[\tfrac{\partial\psi(s,x(s))}{\partial s} + \tfrac{\partial\psi(s,x(s))}{dx}\tfrac{dx(s)}{ds}\right]ds \\
&+ \int_0^1 G_\alpha(|t-s|)\tfrac{\partial m(t,s)}{\partial s}\psi(s,x(s))ds = f(t) + G_\alpha(1-t)m(t,1)\psi(1,x(1))
\end{aligned} \tag{5.8}$$

where $G_\alpha(|s-t|) = \int_0^t g_\alpha(|s-y|)dy$. Now the differentiation of equation (5.8) yields

$$\frac{dx(t)}{dt} - \int_0^1 g_\alpha(|t-s|)m(t,s)\frac{\partial\psi(s,x(s))}{\partial x}\frac{dx(s)}{ds}ds = F(t) \qquad (5.9)$$

where $F(t)$ consists of 8 terms and it is easy to see, term by term, that $h(t)F(t) \in C^{(0,\alpha)}[0,1]$. Multiplying (5.9) by h and letting $x_1 = h(t)\frac{dx(t)}{dt}$, $m_1(t,s) = m(t,s)\frac{\partial\psi(s,x(s))}{\partial x}$ and

$$F_1Z(t) = h(t)F(t) + \int_0^1 g_\alpha(|t-s|)m_1(t,s)(t-s)(1-t-s)\frac{dx(s)}{ds}$$

we find that x_1 satisfies the linear equation

$$x_1(t) - \int_0^1 g_\alpha(|t-s|)m_1(t,s)x_1(s)ds = F_1(t), \qquad 0 \le t \le 1. \qquad (5.10)$$

It is easy to see that $F_1 \in C^{(0.\alpha)}[0,1]$. By Lemma 5.2 (i), $x_1 \in C^{(0,\alpha)}[0,1]$. Let $\hat{x}_1(t) = h(t)\frac{dx_1(t)}{dt}$. A similar analysis to the one above show that $x_1 \in C^{(0,\alpha)}[0,1]$ and $x_1 \in C^1(0,1)$. Noting that $\hat{x}_1(t) = h^2(t)\frac{d^2x(t)}{dt^2} + (1-2t)\frac{dx(t)}{dt}$, we deduce that $h^2(t)\frac{d^2x(t)}{dt^2} \in C^{(0,\alpha)}[0,1]$. Moreover $\frac{dx_1(t)}{dt} = h(t)\frac{dx^2(t)}{dt^2} + (1-2t)\frac{dx(t)}{dt}$. Hence $\frac{d^2x}{dt^2} \in C(0,1)$. This procedure can be repeated to prove that $x \in C^n(0,1)$. This completes the proof. □

In the remainder of this section, we consider the Hammerstein equation with logarithmic singularity because of its important applicability to a class of boundary value problems and its application to the singularity preserving Galerkin scheme that will be discussed in Section 7. We consider

$$x(t) - \int_0^1 \log|t-s|m(t,s)\psi(s,x(s))ds = f(t), \qquad 0 \le t \le 1 \qquad (5.11)$$

(see (1.1) also). With

$$K\Psi x(t) \equiv \int_0^1 \log|t-s|m(t,s)\psi(s,y(s))ds. \qquad (5.12)$$

Then equation (5.11) can be written in operator form as

$$x - K\Psi x = f. \qquad (5.13)$$

We let $W = W_n$ be the linear space spanned by the functions $t^i \log^j t, (1-t)^i \log^j(1-t); i,j = 1,2,...,n-1$. Throughout the remainder of this section, we assume the following conditions:

$$m \in C^{2n}([0,1]\times[0,1]), n \ge 1,\; m \in C^1([0,1]\times[0,1]), n = 0. \qquad (5.14)$$

$$\psi \in C^{2n+1}(R \times R)$$
$$f \in W \oplus W_2^n. \tag{5.15}$$

We define

$$Ky(t) \equiv \int_0^1 \log|t-s| m(t,s) y(s) ds. \tag{5.16}$$

Also let $u_1(t) = t^p \log^q t$, and $u_2(t) = (1-t)^p \log^q(1-t)$, where $p, q \geq 1$ are integers. First we quote the following result (lemma 4.4(2)) from [5].

Lemma 5.5 *Let $f \in W_2^{n-1}$ and assume $m \in C^{n+1}([0,1] \times [0,1])$. Then,*

$$(Kf)(t) = \sum_{j=1}^{n-1} \left[c_j t^j \log t + d_j (1-t)^j \log(1-t)\right] + v_n(t),$$

$$(Ku_1)(t) = \sum_{j=p+1}^{n-1} \sum_{i=1}^{q+1} c_{ij} t^j (\log t)^i + \sum_{j=q+1}^{n-1} d_j (1-t)^j \log(1-t) + v_n(t),$$

and

$$(Ku_2)(t) = \sum_{j=p+1}^{n-1} \sum_{i=1}^{q+1} c_{ij} (1-t)^j (\log(1-t))^i + \sum_{j=q+1}^{n-1} d_j t^j \log t + v_n(t).$$

We also need the following lemmas from [20].

Lemma 5.6 *If $u_1(t) = t^p \log^q t, u_2(t) = (1-t)^r \log^u(1-t)$, where $p, q, r, u \geq 1$ are integers, then $u_1 u_2 \in W \oplus W_2^n$.*

Lemma 5.7 *A product of an W_2^n function with a function in W is in $W_2^n \oplus W$.*

Lemma 5.8 *The operator $K\Psi$ maps $W \oplus W_2^n$ into $W \oplus W_2^{n+1}$.*

The next theorem characterizes the solution of (5.10).

Theorem 5.9 *Suppose the conditions (5.4)-(5.6) hold and x is an isolated solution of (5.1). Then there are constants a_{ij} and b_{ij}, for $i, j = 1, 2, ..., n-1$, and there is a function v_n in W_2^n such that*

$$x(t) = \sum_{i=1}^{n-1} \sum_{j=1}^{n-1} [a_{ij} t^i \log^j t + b_{ij} (1-t)^i \log^j (1-t)] + v_n(t). \tag{5.17}$$

Proof: For $n = 0$, this follows from Lemma 5.8 with $n = 0$. Assume that the result holds for $n = k$, that is, if $f \in W_2^k \oplus W$, then (5.13) holds with $n = k$. Say $x = w_k + v_k$, where $v_k \in W_2^k, w_k = \sum_{i=1}^{k-1} \sum_{j=1}^{k-1} [a_{ij} t^i \log^j t + b_{ij} (1-t)^i \log^j (1-t)]$.

Now consider the case $n = k+1$ and suppose $f \in W_2^{k+1} \oplus W$, here of course $W = W_{k+1}$.

Since $x = w_k + v_k$ we write $x = K\Psi x + f = K\Psi(w_k + v_k) + f$. From Lemma 5.3, $K\Psi(w_k + v_k) \in W \oplus W_2^{k+1}$. The proof is complete. □

6 Gauss-Type Quadarture Schemes

In this section, the Gauss-type quadrature schemes recently developed by Kaneko and Xu [17] are reviewed. The utility of the quadrature schemes becomes apparent in light of the analysis of the previous section concerning the weakly singular Hammerstein equations. Let S be a subset of $[0, 1]$ containing a finite number of points. Define a function ω_S by

$$\omega_S(x) = \inf\{|x - t| : t \in S\}. \tag{6.1}$$

For $\alpha > -1$ and a nonnegative integer k, we say that f belongs to $Type(\alpha, k, S)$ if

$$|f^{(k)}(x) \leq C[\omega_S(x)]^{(\alpha-k)}, \qquad x \notin S, \quad f \in C^k([0,1] \setminus S). \tag{6.2}$$

For $\alpha > 0$, this class of functions was introduced by Rice [43]. For example, $f(x) = x^\alpha$ and $f(x) = \sin(x^\alpha)$ for $\alpha > -1$ belong to $Type(\alpha, k, \{0\})$ for each nonnegative integer k; $f(x) = \log x$ belongs to $Type(0, k, \{0\})$ whereas $f(x) = x \log x$ belongs to $Type(1, k, \{0\})$ for each nonnegative integer k. Now we consider the following integral and the Gauss-type quadrature schemes to approximate it;

$$I(f) \equiv \int_0^1 \rho(x) f(x)\, dx, \tag{6.3}$$

where $f \in Type(\alpha, 2k, S)$ with $\alpha > -1$ and ρ is some weight function.

In particular, for our first case, we take $f \in Type(\alpha, 2k, \{0\}) \cap C[0, 1]$ with $\alpha > 0$ and $\rho \in L_1[0, 1]$. We let $q = \frac{2k}{\alpha}$ and for a positive integer n, let π_α be a partition of $[0, 1]$ given by

$$\begin{aligned} t_0 &= 0 \\ t_j &= (j/n)^q, \qquad j = 1, 2, \ldots, n. \end{aligned} \tag{6.4}$$

On each $[t_i, t_{i+1}]$, $1 \leq i \leq n-1$, let

$$t_i \leq u_1^{(i)} < u_2^{(i)} < \cdots < u_k^{(i)} \leq t_{i+1} \tag{6.5}$$

be the k zeros of the orthogonal polynomial of degree k with respect to the weight function ρ transformed into $[t_i, t_{i+1}]$. The function f is now approximated by the following piecewise polynomial S_k with knots defined by (6.4); $S_k(x)$ is the linear interpolant of f at t_0 and t_1 for $x \in [t_0, t_1]$ and $S_k(x)$ is the Lagrange interpolant of degree $k-1$ to f at $\{u_j^{(i)}\}_{j=1}^k$ for $x \in [t_i, t_{i+1}]$ and for $i = 1, 2, \ldots, n-1$. $I(f)$ is then approximated by $I(S_k)$.

Theorem 6.1 *Let $f \in Type(\alpha, 2k, \{0\}) \cap C[0, 1]$ with $\alpha > 0$ and $\rho \in L_1[0, 1]$ with $\rho > 0$ almost everywhere in $[0, 1]$. Then*

$$|I(f) - I(S_k)| = O(n^{-2k+1}).$$

Proof: Define $E_{k,i}(f) = \int_{t_i}^{t_{i+1}} \rho(x)[f(x) - S_k(x)]\,dx$ for $i = 0, 1, \ldots, n-1$. First,

$$\begin{aligned}
|E_{k,0}(f)| &\le \int_{t_0}^{t_1} \rho(x)|f(x) - S_k(x)|\,dx \\
&\le \int_{t_0}^{t_1} \rho(x)|f(x) - f(0)|dx + \int_{t_0}^{t_1} \rho(x)|S_k(0) - S_k(x)|dx \\
&\le C\int_{t_0}^{t_1} \rho(x)x^\alpha dx + |S_k(0) - S_k(x)| \int_{t_0}^{t_1} \rho(x)dx \\
&\le Ct_1^\alpha + |f(0) - f(t_1)| \int_{t_0}^{t_1} \rho(x)dx \\
&\le Ct_1^\alpha = Cn^{-2k},
\end{aligned}$$

where C denotes a generic constant that is independent of n. For $i = 1, 2, \ldots, n-1$, since $f \in Type(\alpha, 2k, \{0\})$, $f \in C^{2k}[t_i, t_{i+1}]$. Using the well-known error formula for the Gaussian quadrature, there exists $\eta_i \in [t_i, t_{i+1}]$ such that

$$|E_{k,i}(f)| = \frac{|f^{(2k)}(\eta_i)|}{(2k)!} \left| \int_{t_i}^{t_{i+1}} \rho(x)(x - u_1^{(i)})^2 \cdots (x - u_k^{(i)})^2 dx \right|. \qquad (6.6)$$

Assuming without loss of generality that $\alpha \le 2k$, we obtain from (6.6)

$$\begin{aligned}
|E_{k,i}(f)| &\le C|\eta_i|^{\alpha-2k}(t_{i+1} - t_i)^{2k} \le Ct_i^{\alpha-2k}(t_{i+1} - t_i)^{2k} \\
&= C(i^q t_1)^{\alpha-2k}[(i+1)^q - i^q]^{2k} t_1^{2k} \\
&\le Cq^{2k}[(i+1)^{2k(q-1)} i^{q(\alpha-2k)}] t_1^{2k} \\
&\le Cq^{2k} 2^{2k(q-1)} t_1^\alpha \\
&= Cn^{-2k}.
\end{aligned}$$

The second to the last inequality is obtained by noting $(i+1)^q - i^q \le q(i+1)^{q-1}$, whereas the last inequality is obtained by noting $q(\alpha - 2k) = \frac{2k}{\alpha}(\alpha - 2k) = 2k(1-q)$ and

$$(i+1)^{2k(q-1)} i^{q(\alpha-2k)} = \left(\frac{i+1}{i}\right)^{2k(q-1)} \le 2^{2k(q-1)}.$$

Thus

$$|I(f) - I(S_k)| = \left| \sum_{i=0}^{n-1} E_{k,i}(f) \right| \le Cn^{-2k+1}.$$

This completes the proof.□

As the second special case, we consider $f \in Type(\alpha, 2k, \{0\})$ with $\alpha > -1$ and $\rho \in L_\infty[0,1]$. For this case, we construct a piecewise polynomial S_k of degree $k-1$ that approximates f by the following rule; $S_k(x) = 0$ for $x \in [t_0, t_1)$ and $S_k(x)$ is the Lagrange interpolant of degree $k-1$ to f at $\{u_j^{(i)}\}_{j=1}^k$ for $x \in [t_i, t_{i+1})$, $i = 1, 2, \ldots, n-2$ and for $x \in [t_{n-1}, t_n]$ when $i = n-1$. The following theorem can be proved in a similar way to the previous theorem.

Theorem 6.2 *Let* $f \in Type(\alpha, 2k, \{0\})$ *with* $\alpha > -1$ *and* $\rho \in L_\infty[0,1]$ *be positive almost everywhere in* $[0,1]$. *Then*

$$|I(f) - I(S_k)| = O(n^{-2k}).$$

We note that the case for $\rho(x) \equiv 1$ on $[0,1]$ is discussed in detail in [16] under the title of Gauss-Legendre-Type quadrature. It is also demonstrated in [16] that the quadrature is useful in approximating the integrals that must be evaluated in application of a numerical scheme to solve the weakly singular Fredholm equations of the second kind. An application of this quadrature to the weakly singular Hammerstein equation is now discussed. The collocation method for the weakly singular Hammerstein equation is mentioned here. We refer the reader to [17] for a discussion of the Galerkin method for the weakly singular equation.

Due to the type of its singularities at the end points of the solution of the weakly singular Hammerstein equation that were described in Theorem 5.3, one can not expect the collocation method (or the Galerkin method) to produce a numerical solution with optimal convergence rate unless this special regularity property of the solution is taken into account. What Theorem 5.3 reveals in the terminology of the newly defined class of functions $Type(\alpha, k, S)$ is that if x is the solution of equation (1.1) with the kernel k defined by (5.1) and if the forcing function $f \in Type(\beta, k, \{0,1\})$, then $x \in Type(\gamma, k, \{0,1\})$, where $\gamma = \min\{\alpha, \beta\}$. Hence, to approximate x, we use a theorem of Rice from [43]. Namely, when $S_r^\nu(\Pi_n)$ is used as an approximating space, the optimal convergence rate of the collocation solution can be recovered by selecting the partition points by

$$t_i = \begin{cases} \frac{1}{2}\left(\frac{2i}{n}\right)^q, & 0 \le i \le \frac{n}{2}, \\ 1 - t_{n-i}, & \frac{n}{2} < i \le n, \end{cases}$$

where $q \equiv \frac{r}{\gamma}$ is called the index of singularity. Between t_i and t_{i+1} for each i, the collocation points t_{ij} are selected according to (3.14). For convenience, we recall here the collocation equation (3.15).

$$a_{ij} - \sum_{p=0}^{n-1} \int_{t_p}^{t_{p+1}} k(t_{ij}, s)\psi(s, \sum_{k=0}^{r} a_{pk} l_{pk}(s))ds = f(t_{ij}),$$

For $p = i$, $j = 0, 1, \ldots, r$, we need to compute the integrals that contain in their integrands the functions that belong to $Type(\alpha, k, \{t_p, t_{p+1}\})$. The quadrature schemes described in this section now become useful for evaluating these integrals. We note that the examples given in [17], [18] and [19] were obtained by making use of the quadratures described in this section.

7 Singularity Preserving Galerkin Method

In this section, we discuss the method of singularity preserving Galerkin method for equation (1.1) with logarithmic kernel. As in Section 3, define the partition of $[0,1]$ as

$$\Pi_m : 0 = t_0 < t_1 < ... < t_m = 1.$$

Here we assume no condition on the partition points except that with

$$h = \max_{1 \le i \le m} (t_i - t_{i-1}),$$

we assume $h \to 0$ as $m \to \infty$. For convenience of notation, in this section, we denote the space S_r^ν of splines of order r with continuity ν by $S_h^{r,\nu}$. Let d denote the dimension of $S_h^{r,\nu}$ and suppose that $\{B_i\}_{i=1}^d$ is the B-splines basis for $S_h^{r,\nu}$. We let

$$V_h^{r,\nu} \equiv W \oplus S_h^{r,\nu} \tag{7.1}$$

where W is defined in Section 5. We denote the orthogonal projection of $L_2[0,1]$ onto $V_h^{r,\nu}$ by P_h^G. The singularity preserving Galerkin method for approximating the solution of equation (1.1) requires the solution $y_h \in V_h^{r,\nu}$ to satisfy the following equation:

$$y_h - P_h^G K \Psi y_h = P_h^G f. \tag{7.2}$$

More specifically, we need to find y_h in the form

$$y_h(s) = \sum_{i,j=1}^{n-1} \alpha_{ij} s^i \log^j s + \sum_{i,j=1}^{n-1} \beta_{ij} (1-s)^i \log^j (1-s) + \sum_{i=1}^{d} \gamma_i B_i(s) \tag{7.3}$$

where $\{\alpha_{ij}, \beta_{ij}\}_{i,j=1}^{n-1}$ and $\{\gamma_i\}_{i=1}^d$ are found by solving the following system of nonlinear equations:

$$\begin{aligned}
&\textstyle\sum_{i,j=1}^{n-1} \alpha_{ij}(s^i \log^j s, s^p \log^q s) + \sum_{i,j=1}^{n-1} \beta_{ij}((1-s)^i \log^j (1-s), s^p \log^q s) \\
&\qquad + \textstyle\sum_{i=1}^{d} \gamma_i (B_i, s^p \log^q s) - (K\Psi(\sum_{i,j=1}^{n-1} \alpha_{ij} s^i \log^j s \\
&\qquad + \textstyle\sum_{i,j=1}^{n-1} \beta_{ij}(1-s)^i \log^j (1-s) + \sum_{i=1}^{d} \gamma_i B_i), s^p \log^q s) \\
&\qquad\qquad = (f, s^p \log^q s) \qquad p, q = 1, 2, \ldots, n-1
\end{aligned}$$

$$\begin{aligned}
&\textstyle\sum_{i,j=1}^{n-1} \alpha_{ij}(s^i \log^j s, (1-s)^p \log^q (1-s)) + \sum_{i,j=1}^{n-1} \beta_{ij}((1-s)^i \log^j (1-s), \\
&\qquad (1-s)^p \log^q (1-s)) + \textstyle\sum_{i=1}^{d} \gamma_i (B_i, (1-s)^p \log^q (1-s)) \\
&\qquad -(K\Psi(\textstyle\sum_{i,j=1}^{n-1} \alpha_{ij} s^i \log^j s + \sum_{i,j=1}^{n-1} \beta_{ij}(1-s)^i \log^j (1-s) \\
&\qquad + \textstyle\sum_{i=1}^{d} \gamma_i B_i), (1-s)^p \log^q (1-s)) \\
&\qquad\qquad = (f, (1-s)^p \log^q (1-s)) \qquad p, q = 1, 2, \ldots, n-1
\end{aligned}$$

$$\begin{aligned}
&\textstyle\sum_{i,j=1}^{n-1} \alpha_{ij}(s^i \log^j s, B_p) + \sum_{i,j=1}^{n-1} \beta_{ij}((1-s)^i \log^j (1-s), B_p) \\
&\qquad + \textstyle\sum_{i=1}^{d} \gamma_i (B_i, B_p) - (K\Psi(\sum_{i,j=1}^{n-1} \alpha_{ij} s^i \log^j s \\
&\qquad + \textstyle\sum_{i,j=1}^{n-1} b_{i,j}(1-s)^i \log^j (1-s) + \sum_{i=1}^{d} \gamma_i B_i), B_p) \\
&\qquad\qquad = (f, B_p) \qquad p = 1, 2, \ldots, d
\end{aligned}$$

where $(\cdot, \cdot)$ denotes the usual inner product defined on $L_2[0,1]$. Now let P_h be the orthogonal projection of $L_2[0,1]$ onto $S_h^{r,\nu}$. Then we have

$$P_h v \to v \qquad \text{for all } v \in L_2[0,1]. \tag{7.4}$$

It is well known (e.g. [11]) that if $g \in W_2^n$, $n \geq 0$, then for each $h > 0$, there exists $\phi_h \in S_h^{r,\nu}$ such that

$$\|g - \phi_h\|_2 \leq C h^{\min\{n,r\}} \|g\|_{W_2^n}, \tag{7.5}$$

where $C > 0$ is a constant independent of h. By virtue of the fact that $P_h u$ is the best L_2 approximation of u from $S_h^{r,n}$, we see immediately that

$$\|P_h u - u\|_2 \leq \|u - \phi_h\|_2 \leq C h^{\min\{r,n\}} \|u\|_{W_2^n}, \text{ for all } u \in W_2^n. \tag{7.6}$$

The following lemma from [5] is useful in the sequel.

Lemma 7.1 *Let X be a Banach space. Suppose that U_1 and U_2 are two subspaces of X with $U_1 \subseteq U_2$. Assume that $P_1 : X \to U_1$ and $P_2 : X \to U_2$ are linear operators. If P_2 is a projection, then*

$$\|x - P_2 x\|_X \leq (1 + \|P_2\|_X) \|x - P_1 x\|_X \text{ for all } x \in X.$$

For convenience, we introduce operators $\hat{T}$ and T_h by letting

$$\hat{T} y \equiv f + K\Psi y \tag{7.7}$$

and

$$T_h y_h \equiv P_h^G f + P_h^G K\Psi y_h \tag{7.8}$$

so that equations (1.1) and (7.2) can be written respectively as $y = \hat{T}y$ and $y_h = T_h y_h$. The following theorem guarantees the existence of a solution of the singularity preserving Galerkin method (7.2) and describes the accuracy of its approximation.

Theorem 7.2 *Let $y \in L_2[0,1]$ be an isolated solution of equation (1.1). Assume that 1 is not an eigenvalue of the linear operator $(K\Psi)'(y)$, where $(K\Psi)'(y)$ denotes the Fréchet derivative of $K\Psi$ at y. Then the singularity preserving Galerkin approximation equation (7.2) has a unique solution y_h such that $\|y - y_h\|_2 < \delta$ for some $\delta > 0$ and for all $0 < h < h_0$ for some $h_0 > 0$. Moreover, there exists a constant $0 < q < 1$, independent of h, such that*

$$\frac{\alpha_h}{1+q} \leq \|y - y_h\|_2 \leq \frac{\alpha_h}{1-q}, \tag{7.9}$$

where $\alpha_h \equiv \|(I - T_h'(y))^{-1}(T_h(y) - \hat{T}(y))\|_2$. Finally, if $y = w + v$ with $w \in W$ and $v \in W_2^n$, then

$$\|y - y_h\|_2 \leq Ch^{\min\{r,n\}}\|v\|_{W_2^n}, \qquad 0 < h < h_0, \tag{7.10}$$

where $C > 0$ is a constant independent of h.

Proof: The existence of a unique solution y_h of equation (7.2) in the disk of radius δ about y and the inequalities in (7.7) can be proved using Theorem 2 of Vainikko [40]. To get (7.10), first we note from Lemma 7.1, for $v \in L_2[0,1]$,

$$\|P_h^G v - v\|_2 \leq (1 + \|P_h^G\|_2)\|P_h v - v\|_2. \tag{7.11}$$

Now, from (7.9),

$$\begin{aligned} \|y - y_h\|_2 &\leq \tfrac{\alpha_h}{1-q} \\ &= \tfrac{1}{1-q}\|(I - T_h'(y))^{-1}(T_h(y) - \hat{T}(y))\|_2 \\ &\leq C\|P_h^G K\Psi y - K\Psi y + P_h^G f - f\|_2 \\ &= C\|P_h^G y - y\|_2. \end{aligned} \tag{7.12}$$

where C is independent of h. Using the uniform boundedness of $\{P_h^G\}$, (7.11) and (7.12), we obtain

$$\|y - y_h\|_2 \leq Ch^{\min\{r,n\}}\|v\|_{W_2^n}.$$

□

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