A Note on the Finite Element Method with Singular Basis Functions

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Abstract

In this note, we make a few comments concerning the paper of Hughes and Akin [4]. Our primary goal is to demonstrate that the rate of convergence of numerical solutions of the finite element method with singular basis functions depends upon the location of additional collocation points associated with the singular elements.

Key Words: Finite element method with singular basis functions, the collocation method, Singular interpolation elemnts.

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1 Finite Element Method With Singular Basis Functions

In the paper [4], Hughes and Akin made an interesting observation concerning the finite element analysis that incorporates singular element functions. A need for introducing some singular elements as part of basis functions in certain finite element analysis arises out of the following considerations. The solution of certain problems, such as a field problem [1], exhibits highly singular behavior due to geometric features of the spatial domain. On the other hand, in other circumstances, the solution is overwhelmingly affected by the nature of loading and the problem of singularity can be ignored. To satisfy both situations just described, it is thought that an incorporation of singular elements that emulate the solution with the standard polynomial elements may perhaps be desirable. This is the point that was exploited in [4] by Hughes and Akin. In order to make the computations of the finite element method with singular elements more efficient, they consider the following algorithm for constructing interpolation functions. A construction of such algorithm was motivated by the idea that "it is of practical interest to develop techniques for systematically defining shape functions for singularity modeling (and for developing special elements in general), which circumvent the interpolation problem" ([5] p.176). The algorithm that they developed goes as follows:

ALGORITHM Suppose that there are n shape functions N_a, $a = 1, 2, \ldots, n$ which satisfy the interpolation property on the first m nodes r_b, viz., $N_a(r_b) = \delta_{ab}$, $a, b = 1, 2, \ldots, m$ $(m < n)$. Their idea here is to reshape N_a's so that the interpolation property is satisfied on all n nodes. The algorithm is given by

Step 1 $\quad N_{m+1} \leftarrow \frac{N_{m+1}(r) - \sum a=1^m N_{m+1}(r_a) N_a(r)}{N_{m+1}(r_{m+1}) - \sum_{a=1}^{m} N_{m+1}(r_a) N_a(r_{m+1})}$

Step 2 $\quad N_a(r) \leftarrow N_a(r) - N_a(r_{m+1}) N_{m+1}(r), \qquad a = 1, 2, \ldots, m$

Step 3 $\quad$ If $m + 1 < n,$ replace m by $m + 1$ and repeat Steps 1 to 3.
If $m + 1 = n$, stop.

Before we comment on the algorithm, we would like to draw the reader's attention to the recent paper by Dydo and Busby [3]. They observed that "using the algorithm as a subroutine during the finite element shape function generation requires re-evaluation of all terms of the shape functions at each nodal point and eventually leads to excessive round-off error". To prevent this difficulty, they develop an equivalent algorithm in matrix form to generate finite elements with special properties. The results reported in [3] demonstrate a considerable increase in computational efficiency.

We now return to our discussion of the algorithm. To demonstrate this algorithm, we borrow one of the examples from [5]. Let $r_1 = 0$, $r_2 = \frac{1}{2}$ and $r_3 = 1$. The shape functions that we reconstruct are $N_1(r) = 1 - 2r$, $N_2(r) = 2r$ and $N_3(r) = r^\alpha$ where α representing some real number. Note that $N_a(r_b) = \delta_{ab}$, $1 \leq a, b \leq 2$. An application of the above algorithm gives

$$\begin{aligned} N_1(r) &\leftarrow 1 - 2r + \left[\tfrac{r^\alpha - 2(\frac{1}{2})^\alpha r}{1 - 2(\frac{1}{2})^\alpha}\right] \\ N_2(r) &\leftarrow 2r - 2\left[\tfrac{r^\alpha - 2(\frac{1}{2})^\alpha r}{1 - 2(\frac{1}{2})^\alpha}\right] \\ N_3(r) &\leftarrow \tfrac{r^\alpha - 2(\frac{1}{2})^\alpha r}{1 - 2(\frac{1}{2})^\alpha}. \end{aligned}$$

Of course, the newly defined shape functions satisfy

$$N_a(r_b) = \delta_{ab} \qquad 1 \leq a, b \leq 3. \tag{1}$$

It is important to note at this point that the algorithm is subject to the location of the interpolation points r_b, $m+1 \leq b \leq n$. Clearly, step 1 of algorithm is sensitive to the location. Namely, if these interpolation points are such that

$$N_{m+1}(r_{m+1}) - \sum_{a=1}^{m} N_{m+1}(r_a) N_a(r_{m+1}) = 0 \tag{2}$$

then the algorithm does not work. Out of this observation, there seems to arise a profound and difficult problem in the area of approximation theory. The problem is important in that the success of the finite element method using the collocation scheme with singular basis functions hinges on a resolution of this problem. To describe it, let W_p^k denote the Sobolev space,

$$W_p^k = \{f | f^{(k)} \in L_p(\Omega)\}$$

where $f^{(k)}$ denote the kth generalized derivative and Ω is a bounded region in R. The theory extends easily to higher dimensions. Now let $U \equiv \text{span}[N_a]_{a=m+1}^n$ where $\{N_a\}_{a=m+1}^n$ represent the singular elements. Also denote by S_h^k the approximation space spanned by $\{N_a\}_{a=1}^m$. Here S_h^k is usually taken as the space of piecewise polynomials of degree $k - 1$ with length of each subinterval h. Our goal is to approximate each element of $U \oplus W_p^k$ by an element from $U \oplus S_h^k$ by interpolation. That is, if $\{r_b\}_{b=1}^m$ is such that $N_a(r_b) = \delta_{ab}$ for $1 \leq a, b \leq m$, then for each $f \in U \oplus W_p^k$, we must find $v \in U \oplus S_h^k$ that satisfies

$$v(r_b) = f(r_b) \qquad b = 1, \ldots, n, \tag{3}$$

where $\{r_b\}_{b=m+1}^{n}$ are specified interpolation points for $\{N_a\}_{a=m+1}^{n}$. Denote the interpolation projector of $U \oplus W_p^k$ to $U \oplus S_h^k$ by P_h. Namely P_h is defined so that

$$P_h f(s) = v(s) \qquad s \in \Omega. \tag{4}$$

Notice that $P_h^2 = P_h$. In order to achieve a convergence by the collocation scheme in the finite element method, we must examine the following inequality. Here we assume that the number of singular basis functions, $n-m$, is fixed.

$$|\det[N_a(r_b)]|_{a,b=1}^{n} \geq \epsilon > 0 \qquad \text{for all } n. \tag{5}$$

This inequality is a necessary and sufficient condition for the algorithm of Hughes and Akin to work. It is important to remark at this point that the success of algorithm depends upon the existence of a solution to the interpolation problem (3) which in turn is equivalent to the condition (5). For each fixed index n (hence for fixed m) , it is not difficult to find $n-m$ interpolation points r_b, $m+1 \leq b \leq n$, that correspond to the singular basis elements N_a, $m+1 \leq a \leq n$ for which the inequality in (5) is satisfied. What is difficult here is the question of locating $n-m$ points for as many singular elements for which condition (5) is satisfied for all n. The problem of finding $n-m$ interpolation points for singular basis functions that work for all n is currently under investigations. In the following section, we proceed our discussion of the interpolation problem which leads to a finite element analysis with singular elements. The discussion will provide information concerning the rate of approximation of interpoalation. Interpolation examples at the end of the section show that rates of convergence are quite sensitive to locations of interpolation points for singular elements.

2 Convergence Analysis

When condition (5) is satistied, one can deduce the rate of convergence of the projector P_h to the identity operator I. As is well known -e.g., [5], the convergence rate of such interpolation projectors determine the rate of convergence of the finite element method that uses collocation scheme. The following theorem of Cao and Xu [2] is useful. We sketch a proof for completeness.

Lemma 2.1 *Let X be a Banach space. Assume that U_1 and U_2 are two subspaces of X with $U_1 \subseteq U_2$. Moreover assume that $P_1: X \to U_1$ and $P_2: X \to U_2$ are linear operators. If P_2 is a projection, then*

$$\|x - P_2 x\|_X \leq (1 + \|P_2\|_X)\|x - P_1 x\|_X \qquad \textit{for all } x \in X.$$

Proof: Let $x \in X$. We write

$$x - P_2 x = (x - P_1 x) + (P_1 x - P_2 x).$$

Since $P_1 x \in U_1$ and $U_1 \subseteq U_2$, we have $P_2 P_1 x = P_1 x$. Hence,

$$\begin{aligned} x - P_2 x &= x - P_1 x + P_2 P_1 x - P_2 x \\ &= (I - P_2)(x - P_1 x). \end{aligned}$$

It follows that

$$\|x - P_2 x\|_X \leq (1 + \|P_2\|_X)\|x - P_1 x\|_X \ for\ all\ x \in X.$$

□

This lemma seems to suggest a reason for which the method of interpolation using a set of singular elements is sensitive to the location of interpolation points corresponding to the singular basis functions. We let P_1 be the interpolatory projection of $U \oplus W_p^k$ onto S_h^k, namely for each $f \in U \oplus W_p^k$, $P_1 f \in S_h^k$ is defined by

$$P_1 f(r_a) = f(r_a), \qquad a = 1, \ldots, m.$$

Theorem 2.2 *Assume that $y = u + v$ with $u \in U$ and $v \in W_p^k$. Let P_h be the projection defined by (4). Then*

$$\|P_h y - y\|_p \leq (1 + \|P_h\|_p)\|y - P_1 y\|_p$$

where $1 \leq p \leq \infty$.

We make two observations here. First, condition (5) plays the essential role for the pointwise convergence of P_h to the identity operator I, which, in turn, by the uniform bounded principle, guarantees the uniform boundedness of $\|P_h\|$ for $h > 0$. Second, because of a singular component present in y, it is expected that $\|y - P_1 y\|_p$ does not converge at the optimal rate. Two observations seem to provide some explanations to the phenomenon revealed in the example below. A comprehensive study of the power of approximations by splines can be found in [8].

A similar study of incorporating the singular elements into an approximating basis was given in the numerical analysis for the weakly singular Fredholm equations. The results from [2] were recently generalized to a class of nonlinear Hammerstein equations in [6], which in turn extends the work in [7]. These papers are concerned with the singularity preserving Galerkin methods and the analysis associated with these methods appear more straightforward. We close this note by demonstrating the sensitivity of the location of interpolation points for singular elements.

EXAMPLE: Let $f(x) = \sqrt{x} + \sqrt{1-x} + x^2$. We wish to approximate f over $[0,1]$ by an element from $U \oplus S_h^2$, where $U = \text{span}[\sqrt{x}, \sqrt{1-x}]$. Let $\{x_i\}_{i=0}^n$ be the uniform partition of $[0,1]$ defined by $x_i = \frac{i}{n}$, $i = 0, 1, \ldots, n$ and $h = \frac{1}{n}$. The interpolation points used to define an element from S_h^2 are taken to be the zeros of the second degree Legendre polynomials transformed into $[x_{i-1}, x_i]$ for $i = 1, 2, \ldots, n$. The following data shows that (a) when the interpolation points for the singular elements are taken to be $t_1 = \frac{1}{5}$ and $t_2 = \frac{4}{5}$ for each n, the convergence is $O(h^{1/2})$, whereas (b) when $t_1 = \frac{1}{2^n}$ and $t_2 = 1 - t_1$, then the convergence is of the order $O(h^2)$.

interpolation point $t_1 =$	$1/2^n$	$1/5$
n = 4	0.0513168	0.0513168
n = 6	0.0008437	0.0235540
convergence exponent =	1.99	0.56

Table 1. Error and convergence rate data for the example

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