



On Taylor-series expansion methods for the second kind integral equations

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ABSTRACT

In this paper, we comment on the recent papers by Yuhe Ren et al. (1999) [1] and Maleknejad et al. (2006) [7] concerning the use of the Taylor series to approximate a solution of the Fredholm integral equation of the second kind as well as a solution of a system of Fredholm equations. The technique presented in Yuhe Ren et al. (1999) [1] takes advantage of a rapidly decaying convolution kernel $k(|s-t|)$ as $|s-t|$ increases. However, it does not apply to equations having other types of kernels. We present in this paper a more general Taylor expansion method which can be applied to approximate a solution of the Fredholm equation having a smooth kernel. Also, it is shown that when the new method is applied to the Fredholm equation with a rapidly decaying kernel, it provides more accurate results than the method in Yuhe Ren et al. (1999) [1]. We also discuss an application of the new Taylor-series method to a system of Fredholm integral equations of the second kind.

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1. Introduction

In paper [1], a Taylor-series expansion method to approximate a solution of a class of Fredholm integral equations of the second kind was considered. The Fredholm equation of the second kind takes the following form:

$$x(s) - \int_0^1 k(s, t)x(t)dt = y(s), \quad 0 \leq s \leq 1, \quad (1.1)$$

where it is assumed that 1 is not the eigenvalue of the operator

$$Tx(s) \equiv \int_0^1 k(s, t)x(t)dt.$$

The kernel $k(s, t) = k(|s - t|)$ is assumed to be continuous in $I \equiv [0, 1]$ and decreases as $|s - t|$ increases from zero or $k(s, t) = a(s, t)\kappa(s - t)$ with a is continuous for $s, t \in I$ and $\kappa(s - t) = O(|s - t|^{-\alpha})$, $0 < \alpha < 1$. Our numerical experiments indicate that the Taylor method introduced in [1] is effective under the first assumption, particularly, in the case that the rate of decay to 0 of $k(|s - t|)$ is sufficiently large, but as was reported in [1], the technique does not perform well under the second assumption of weakly singular kernel. For numerical solutions of weakly singular Fredholm equations, the present authors suggest that it is better to use the standard Galerkin method or the collocation method to obtain numerical solutions which exhibit optimal order of convergence; see, e.g., [2–5] and the references cited within. As described in [1], the Fredholm equations of the second kind play an important role in many physical applications which include potential theory and Dirichlet problems, particle transport problems of astrophysics and radiative heat transfer problems. A reader may consult the references provided in [1] for these applied problems.

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The first step in the Taylor expansion method presented in [1] is to write

$$x(t) \approx x(s) + x'(s)(t-s) + \cdots + \frac{1}{n!} x^{(n)}(s)(t-s)^n. \quad (1.2)$$

Substituting (1.2) for $x(t)$ in the integral in (1.1), we obtain

$$\begin{aligned} & \left[1 - \int_0^1 k(s, t) dt \right] x(s) - \left[\int_0^1 k(s, t)(t-s) dt \right] x'(s) - \cdots \\ & - \left[\frac{1}{n!} \int_0^1 k(s, t)(t-s)^n dt \right] x^{(n)}(s) \approx y(s), \quad 0 < s < 1. \end{aligned} \quad (1.3)$$

Eq. (1.3) represents an n th order linear ordinary differential equation with variable coefficients. However, in order to carry out the solution process, it is necessary that an appropriate number of boundary conditions be introduced. These boundary conditions may be found from an actual experiment, but this is not always possible. To circumvent this problem, paper [1] proceeds as follows. First, differentiating (1.1) n times, one obtains

$$\begin{aligned} x'(s) - \int_0^1 k'_s(s, t)x(t)dt &= y'(s) \\ &\vdots \\ x^{(n)}(s) - \int_0^1 k_s^{(n)}(s, t)x(t)dt &= y^{(n)}(s), \end{aligned} \quad (1.4)$$

where $k_s^{(i)}(s, t) = \partial^{(i)} k(s, t) / \partial s^i$, $i = 1, \dots, n$. Next, $x(t)$ is replaced by $x(s)$ to obtain, for $0 < s < 1$,

$$\begin{aligned} x'(s) - \int_0^1 k'_s(s, t)dt x(s) &= y'(s) \\ &\vdots \\ x^{(n)}(s) - \int_0^1 k_s^{(n)}(s, t)dt x(s) &= y^{(n)}(s). \end{aligned} \quad (1.5)$$

The step taken in (1.5) characterizes the Taylor technique of [1] and it is justified by the first assumption above that $k(|s-t|)$ decays rapidly as $|s-t|$ increases. Eq. (1.3) together with equations in (1.5) can be used to solve for $x, x', \dots, x^{(n)}$. More specifically, we solve the following system of linear equations for $x(s), x'(s), \dots, x^{(n)}(s)$.

$$\begin{bmatrix} 1 - \int_0^1 k(s, t)dt & - \int_0^1 k(s, t)(t-s)dt & \cdots & - \frac{1}{n!} \int_0^1 k(s, t)(t-s)^n dt \\ - \int_0^1 k'_s(s, t)dt & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ - \int_0^1 k_s^{(n)}(s, t)dt & 0 & \cdots & 1 \end{bmatrix} \begin{bmatrix} x(s) \\ x'(s) \\ \vdots \\ x^{(n)}(s) \end{bmatrix} = \begin{bmatrix} y(s) \\ y'(s) \\ \vdots \\ y^{(n)}(s) \end{bmatrix}. \quad (1.6)$$

A step taken in (1.5) needs some investigation. In (1.1), solution x is expanded by the Taylor series to its n th degree and yet in (1.5) $x(t)$ is replaced by its constant approximation $x(s)$. Despite this, under a fast decaying kernel, solution of (1.6) produces reasonably accurate approximation as reported in the paper [1]. In the next section, we propose to retain the n th degree expansion of Taylor series for $x(t)$ in (1.5). Of course, this requires additional computations, but the additional cost is justified by the fact that the new method applies not only to a much wider class of Fredholm equations but also produces more accurate approximations. Also, the present method computes $x(s), x'(s), \dots, x^{(n)}(s)$ all within the same accuracy in the same solution process.

2. Modified Taylor-series method

Here, we begin by replacing each $x(t)$ in (1.5) by the right side of (1.2) to obtain

$$\begin{aligned} & - \int_0^1 k'_s(s, t)dt x(s) - \left[\int_0^1 k'_s(s, t)(t-s)dt - 1 \right] x'(s) - \cdots - \frac{1}{n!} \int_0^1 k'_s(s, t)(t-s)^n dt x^{(n)}(s) = y'(s) \\ & \vdots \\ & - \int_0^1 k_s^{(n)}(s, t)dt x(s) - \int_0^1 k_s^{(n)}(s, t)(t-s)dt x'(s) - \cdots - \left[\frac{1}{n!} \int_0^1 k_s^{(n)}(s, t)(t-s)^n dt - 1 \right] x^{(n)}(s) = y^{(n)}(s). \end{aligned} \quad (2.1)$$

Combining (2.1) with (1.3), we obtain

$$\begin{bmatrix} 1 - \int_0^1 k(s, t) dt & - \int_0^1 k(s, t)(t-s) dt & \cdots & - \frac{1}{n!} \int_0^1 k(s, t)(t-s)^n dt \\ - \int_0^1 k'_s(s, t) dt & 1 - \int_0^1 k'_s(s, t)(t-s) dt & \cdots & - \frac{1}{n!} \int_0^1 k'_s(s, t)(t-s)^n dt \\ \vdots & \vdots & \ddots & \vdots \\ - \int_0^1 k_s^{(n)}(s, t) dt & - \int_0^1 k_s^{(n)}(s, t)(t-s) dt & \cdots & 1 - \frac{1}{n!} \int_0^1 k_s^{(n)}(s, t)(t-s)^n dt \end{bmatrix} \begin{bmatrix} x(s) \\ x'(s) \\ \vdots \\ x^{(n)}(s) \end{bmatrix} = \begin{bmatrix} y(s) \\ y'(s) \\ \vdots \\ y^{(n)}(s) \end{bmatrix}. \quad (2.2)$$

Eqs. (1.6) and (2.2) delineate the difference between the two methods. In [1], all integrals in (2.2) of the form $\frac{1}{j!} \int_0^1 k_s^{(j)}(s, t)(t-s)^j dt$, with $i, j \geq 1$ are set to zero which is justified by a rapidly decaying convolution kernel $k(|s-t|)$ as $|s-t|$ increases. Therefore, the technique in [1] only applies to this case and the accuracy of numerical solution depends critically upon this rate of decay of each kernel. Also as the next simple example shows, the present method is applicable to Fredholm equations with non-convolution kernels, whereas the method in [1] is not.

Example 2.1. We consider (1.1) with $k(s, t) = st$ and $y(s) = \frac{2}{3}s$ so that exact solution is $x(s) = s$. Let $n = 1$ in (2.2), i.e.,

$$\begin{bmatrix} 1 - \int_0^1 k(s, t) dt & - \int_0^1 k(s, t)(t-s) dt \\ - \int_0^1 k'_s(s, t) dt & 1 - \int_0^1 k'_s(s, t)(t-s) dt \end{bmatrix} \begin{bmatrix} x(s) \\ x'(s) \end{bmatrix} = \begin{bmatrix} y(s) \\ y'(s) \end{bmatrix}. \quad (2.3)$$

Substituting the specific forms for k and y , Eq. (2.3) produces the exact solution $x(s) = s$, but the solution of (1.6) diverts from the actual solution significantly. The kernel $k(s, t) = st$ above was selected to demonstrate the difference of the two methods, but a similar kernel, which is a tensor product of univariate functions in s and t , arises in a reformulation of the two-point boundary value problem;

$$\begin{aligned} \frac{d^2 x}{dt^2} &= \lambda x, \quad a < t < b \\ x(a) &= 0 \\ x(b) &= 0. \end{aligned}$$

This reduces to the following homogeneous Fredholm equation of the second kind,

$$x(s) = \lambda \int_a^b k(s, t)x(t) dt$$

where

$$k(s, t) = \begin{cases} \frac{(s-b)(t-a)}{(b-a)}, & a \leq t \leq s \leq b \\ \frac{(s-a)(t-b)}{(b-a)}, & a \leq s \leq t \leq b. \end{cases}$$

The homogeneous Fredholm equation can be solved by transforming it to

$$x(s) = \lambda \int_a^s k(s, t)x(t) dt + \lambda \int_s^b k(s, t)x(t) dt,$$

and applying the Taylor-series method which was developed in [6].

In order to analyze the error term of the current method, substituting $x(t)$ in (1.1) by its Taylor series, we obtain

$$x(s) - \int_0^1 k(s, t) \left[\sum_{r=0}^n \frac{x^{(r)}(s)}{r!} (t-s)^r + \frac{x^{(n+1)}(\xi(s))}{(n+1)!} (t-s)^{n+1} \right] dt = y(s), \quad 0 \leq s \leq 1. \quad (2.4)$$

Also, if the solutions of (2.2) are denoted by $\bar{x}(s)$, $\bar{x}'(s)$, $\dots$, $\bar{x}^{(n)}(s)$, then they satisfy the first equation of (2.2) which is

$$\bar{x}(s) - \int_0^1 k(s, t) \sum_{r=0}^n \frac{\bar{x}^{(r)}(s)}{r!} (t-s)^r dt = y(s), \quad 0 \leq s \leq 1. \quad (2.5)$$

From (2.4) and (2.5),

Table 1Numerical approximation for $x(s)$ in Example 2.2 with $n = 5$.

s	$x(s)$			Absolute error	
	Exact	Our approx.	Ren's approx. [1]	Our approx.	Ren's approx. [1]
0.0	1.00000	1.00000	0.99890	1.0536e–009	1.1023e–003
0.1	1.01001	1.01001	1.00920	6.7344e–009	8.0899e–004
0.2	1.04032	1.04032	1.03981	1.6373e–008	5.1255e–004
0.3	1.09243	1.09243	1.09218	3.2297e–008	2.4880e–004
0.4	1.17024	1.17024	1.17020	3.9788e–008	4.1831e–005
0.5	1.28125	1.28125	1.28135	3.1855e–008	1.0193e–004
0.6	1.43776	1.43776	1.43796	7.0708e–009	2.0148e–004
0.7	1.65807	1.65807	1.65838	3.4706e–008	3.1374e–004
0.8	1.96768	1.96768	1.96823	9.6291e–008	5.5126e–004
0.9	2.40049	2.40049	2.40159	1.6442e–007	1.1021e–003
1.0	3.00000	3.00000	3.00224	9.0310e–008	2.2437e–003

$$[x(s) - \bar{x}(s)] - \int_0^1 k(s, t) \sum_{r=0}^n \frac{[x^{(r)}(s) - \bar{x}^{(r)}(s)]}{r!} (t-s)^r = \frac{x^{(n+1)}(\xi(s))}{(n+1)!} \int_0^1 k(s, t) (t-s)^{n+1} dt.$$

Proceeding similarly for the remaining equations in (2.2), the errors $x^{(r)}(s) - \bar{x}^{(r)}(s)$, $r = 0, 1, \dots, n$ can be computed by solving

$$\begin{bmatrix}
 1 - \int_0^1 k(s, t) dt & - \int_0^1 k(s, t)(t-s) dt & \cdots & - \frac{1}{n!} \int_0^1 k(s, t)(t-s)^n dt \\
 - \int_0^1 k'_s(s, t) dt & 1 - \int_0^1 k'_s(s, t)(t-s) dt & \cdots & - \frac{1}{n!} \int_0^1 k'_s(s, t)(t-s)^n dt \\
 \vdots & \vdots & \ddots & \vdots \\
 - \int_0^1 k_s^{(n)}(s, t) dt & - \int_0^1 k_s^{(n)}(s, t)(t-s) dt & \cdots & 1 - \frac{1}{n!} \int_0^1 k_s^{(n)}(s, t)(t-s)^n dt
 \end{bmatrix}
 \begin{bmatrix}
 x(s) - \bar{x}(s) \\
 \vdots \\
 \vdots \\
 x^{(n)}(s) - \bar{x}^{(n)}(s)
 \end{bmatrix}
 =
 \begin{bmatrix}
 \frac{x^{(n+1)}(\xi(s))}{(n+1)!} \int_0^1 k(s, t)(t-s)^{n+1} dt \\
 \vdots \\
 \vdots \\
 \frac{x^{(n+1)}(\xi(s))}{(n+1)!} \int_0^1 k_s^{(n)}(s, t)(t-s)^{n+1} dt
 \end{bmatrix}. \quad (2.6)$$

Denote (2.6) by $AE = F$ so that the vector E of errors can be bounded as

$$\|E\| \leq \|A^{-1}\| \|F\|, \quad (2.7)$$

here $\|\cdot\|$ denotes a vector norm and its corresponding matrix norm.

System of Eq. (2.6) shows that the present method computes the solution of (1.1) exactly if the solution is a polynomial of degree n or less and computations are carried out exactly. Example 2.1 above demonstrates this point. Eq. (2.6) also reveals that if $x \in C^\infty[0, 1]$ and $\max_{0 \leq s \leq 1} \left| \int_0^1 k_s^{(j)}(s, t)(t-s)^{n+1} dt \right| \leq C$ for all $n = 0, 1, \dots$ and $0 \leq j \leq n+1$ with $C > 0$, then the current method converges as $n \rightarrow \infty$. Throughout our numerical experiments, we found the method to be stable relative to the choice of n and to provide accurate approximate solutions.

As an additional example, we solve Example 2 of [1] and compare the results between the two methods.

Example 2.2. Consider

$$x(s) - \frac{2}{\pi} \int_0^1 k(s, t)x(t)dt = y(s), \quad 0 \leq s \leq 1,$$

where $k(s, t) = [4 + (s-t)^2]^{-1}$ and $y(s)$ is chosen so that $x(s) = 1 + s^2 + s^5$ is the solution. Note that when $y(s) = 1$, this represents the well known Love's equation which arises in electrostatics. Numerical results with $n = 5$ gave excellent approximations for x as well as for its derivatives. The numerical approximation for $x(s)$ and its first derivative with $n = 5$ are shown in Tables 1 and 2, respectively. For each value of s , we recorded computational time spent to obtain the result. The average elapsed time using our technique is 0.05678 s while the average elapsed time using Ren's technique is 0.03253 s.

Table 2Numerical approximation for $x'(s)$ with $n = 5$ for Example 2.2.

s	$x'(s)$			Absolute error	
	Exact	Our approx.	Ren's approx. [1]	Our approx.	Ren's approx. [1]
0.0	0.00000	0.00000	−0.02277	1.7018e−008	2.2772e−002
0.1	0.20050	0.20050	0.17962	2.2075e−008	2.0875e−002
0.2	0.40800	0.40800	0.38974	2.0652e−008	1.8261e−002
0.3	0.64050	0.64050	0.62517	5.9138e−008	1.5327e−002
0.4	0.92800	0.92800	0.91545	5.9923e−008	1.2548e−002
0.5	1.31250	1.31250	1.30197	1.4485e−007	1.0534e−002
0.6	1.84800	1.84800	1.83788	2.1443e−007	1.0118e−002
0.7	2.60050	2.60050	2.58801	1.8399e−007	1.2487e−002
0.8	3.64800	3.64800	3.62870	2.4116e−007	1.9304e−002
0.9	5.08050	5.08050	5.04769	1.7642e−007	3.2807e−002
1.0	7.00000	7.00000	6.94413	6.4720e−008	5.5867e−002

3. System of Fredholm integral equations

In this section, we apply the modified Taylor-series method to a system of Fredholm integral equations of the second kind. In a recent paper [7], the technique in [1] was extended to a system of Fredholm equations. Following the paper [1], authors of the paper [7] also replace, in each of the derivative equations, $x_j^{(r)}(t)$ by $x_j^{(r)}(s)$. Again the success of this technique depends heavily upon rapidly decaying kernels $k_{ij}(|s - t|)$ for each i, j as $|s - t|$ increases. We note that Examples 1 and 2 of [7] do not satisfy this critical condition of decaying convolution kernel. The kernels in Example 1 are of convolution type but do not decay to zero as $|s - t|$ increases, whereas the kernels in Example 2 are not of convolution type. This is more likely the explanation of why the accuracies of approximations reported in [7] are not high despite high order derivatives taken in their computations. We will demonstrate in Examples 3.1 and 3.2 below that the exact solutions of Example 1 in [7] can be found by the current method and far more accurate solutions can be obtained for Example 2 in [7].

Consider a system of Fredholm equations of the second kind,

$$\mathbf{X}(s) - \int_0^1 \mathbf{K}(s, t) \mathbf{X}(t) dt = \mathbf{Y}(s), \quad 0 \leq s \leq 1, \quad (3.1)$$

where

$$\mathbf{X}(s) = [x_1(s), x_2(s), \dots, x_m(s)]^T,$$

$$\mathbf{Y}(s) = [y_1(s), y_2(s), \dots, y_m(s)]^T,$$

$$\mathbf{K}(s, t) = [k_{ij}(s, t)], \quad i, j = 1, 2, \dots, m.$$

The i th equation of (3.1) is given by

$$x_i(s) - \int_0^1 \sum_{j=1}^m k_{ij}(s, t) x_j(t) dt = y_i(s), \quad i = 1, 2, \dots, m. \quad (3.2)$$

Since

$$x_j(t) \approx x_j(s) + x_j'(s)(t - s) + \dots + \frac{1}{n!} x_j^{(n)}(s)(t - s)^n \quad (3.3)$$

substituting (3.3) into (3.2), we obtain

$$x_i(s) - \sum_{r=0}^n \sum_{j=1}^m \frac{1}{r!} \left[\int_0^1 k_{ij}(s, t)(t - s)^r dt \right] x_j^{(r)}(s) \approx y_i(s), \quad i = 1, 2, \dots, m. \quad (3.4)$$

Differentiating (3.2) n times,

$$x_i^{(l)}(s) - \int_0^1 \sum_{j=1}^m k_{ij}^{(l)}(s, t) x_j(t) dt = y_i^{(l)}(s), \quad i = 1, 2, \dots, m, \quad l = 1, \dots, n. \quad (3.5)$$

Substituting (3.3) this time into (3.5), we get

$$x_i^{(l)}(s) - \sum_{r=0}^n \sum_{j=1}^m \frac{1}{r!} \left[\int_0^1 k_{ij}^{(l)}(s, t)(t - s)^r dt \right] x_j^{(r)}(s) \approx y_i^{(l)}(s), \quad i = 1, 2, \dots, m, \quad l = 1, \dots, n. \quad (3.6)$$

Eqs. (3.4) and (3.6) represent a system of $(n + 1)m$ equations in that many unknown functions, $\{x_i^{(l)}(s)\}$, $i = 1, \dots, m$; $l = 0, 1, \dots, n$. It should be pointed out that in [7], following the method established in [1], Eq. (3.5) are replaced by

Table 3Numerical results for Example 3.1 with $n = 5$. The average elapsed time is 0.17524 s.

s	$x_1(s)$			$x_2(s)$		
	Exact	Approximate	Absolute error	Exact	Approximate	Absolute error
0.0	0.00000	0.00000	2.2459e−016	0.00000	−0.00000	4.5066e−016
0.1	0.01000	0.01000	1.0408e−017	−0.08900	−0.08900	2.7756e−017
0.2	0.04000	0.04000	2.0817e−017	−0.15200	−0.15200	2.7756e−017
0.3	0.09000	0.09000	4.1633e−017	−0.18300	−0.18300	2.7756e−017
0.4	0.16000	0.16000	8.3267e−017	−0.17600	−0.17600	8.3267e−017
0.5	0.25000	0.25000	5.5511e−017	−0.12500	−0.12500	1.3878e−016
0.6	0.36000	0.36000	5.5511e−017	−0.02400	−0.02400	2.3592e−016
0.7	0.49000	0.49000	0.0000e+000	0.13300	0.13300	1.3878e−016
0.8	0.64000	0.64000	0.0000e+000	0.35200	0.35200	5.5511e−017
0.9	0.81000	0.81000	5.5511e−016	0.63900	0.63900	8.8818e−016
1.0	1.00000	1.00000	8.8818e−016	1.00000	1.00000	8.8818e−016

Table 4Numerical results for Example 3.2 with $n = 5$. The average elapsed time is 0.16950 s.

s	$x_1(s)$			$x_2(s)$		
	Exact	Approximate	Absolute error	Exact	Approximate	Absolute error
0.1	0.10000	0.10001	6.2649e−006	0.99500	0.99496	4.2641e−005
0.2	0.20000	0.20000	7.1104e−007	0.98007	0.98005	1.9838e−005
0.3	0.30000	0.30000	5.1717e−007	0.95534	0.95533	8.1195e−006
0.4	0.40000	0.40000	4.3203e−007	0.92106	0.92106	2.8836e−006
0.5	0.50000	0.50000	1.3852e−007	0.87758	0.87758	1.1828e−006
0.6	0.60000	0.60000	1.6692e−007	0.82534	0.82533	1.8694e−006
0.7	0.70000	0.70000	6.6413e−007	0.76484	0.76484	5.7789e−006
0.8	0.80000	0.80000	1.7281e−006	0.69671	0.69669	1.6118e−005
0.9	0.90000	0.90000	3.7858e−006	0.62161	0.62157	3.8865e−005
1.0	1.00000	1.00001	7.2151e−006	0.54030	0.54022	8.3172e−005

$$x_i^{(l)}(s) - \int_0^1 \sum_{j=1}^m k_{ij}^{(l)}(s, t) dt x_j(s) = y_i^{(l)}(s), \quad i = 1, 2, \dots, m, \quad l = 1, \dots, n. \quad (3.7)$$

Before we discuss numerical examples, we note that when the method in [7] is described between equations (7) and (14) in Section 2, only the first derivatives are taken in (13). Since the second order Taylor expansion is used in equation (12), it seems to the present authors that the second derivatives must be computed. Otherwise, we have an under-determined system to solve for $f_1, f_2, f'_1, f'_2, f''_1$ and f''_2 .

Example 3.1. This is Example 1 of [7]. Consider the following Fredholm system of integral equation

$$\begin{cases} x_1(s) - \int_0^1 (s-t)^3 x_1(t) dt - \int_0^1 (s-t)^2 x_2(t) dt = y_1(s), \\ x_2(s) - \int_0^1 (s-t)^4 x_1(t) dt - \int_0^1 (s-t)^3 x_2(t) dt = y_2(s), \end{cases} \quad (3.8)$$

with $y_1(s) = \frac{1}{20} - \frac{11}{30}s + \frac{5}{3}s^2 - \frac{1}{3}s^3$ and $y_2(s) = -\frac{1}{30} - \frac{41}{60}s + \frac{3}{20}s^2 + \frac{23}{12}s^3 - \frac{1}{3}s^4$. The exact solutions are $x_1(s) = s^2$ and $x_2(s) = -s + s^2 + s^3$. The numerical solutions using (3.4) and (3.6) with $n = 5$ are shown in Table 3. It is noted that, even though Table 3 is presented with $n = 5$ in agreement with other tables in this paper, $n = 4$ is sufficient to attain the same accuracy since the solutions $x_1(s)$ and $x_2(s)$ are polynomials of degree 2 and 3 respectively and kernels are also polynomials and their degrees are less than or equal to 4. The situation is similar to Example 2.1 in which the exact solution was obtained.

Example 3.2. This is Example 2 of [7]. Consider the following Fredholm system of integral equation

$$\begin{cases} x_1(s) + \int_0^1 t \cos s x_1(t) dt + \int_0^1 s \sin t x_2(t) dt = y_1(s), \\ x_2(s) + \int_0^1 e^{st^2} x_1(t) dt + \int_0^1 (s+t) x_2(t) dt = y_2(s), \end{cases} \quad (3.9)$$

with $y_1(s) = \frac{\cos s}{3} + \frac{s \sin^2 1}{2} + s$ and $y_2(s) = \frac{e^s - 1}{2s} + \cos s + (s+1) \sin 1 + \cos 1 - 1$. The exact solutions are $x_1(s) = s$ and $x_2(s) = \cos s$. The numerical solutions with $n = 5$ are shown in Table 4.

Table 4 demonstrates much improved approximations compared to those reported in [7].

4. Conclusion

In this paper, a new Taylor-series method for approximating a solution of the Fredholm equation of the second kind was presented. The method delivers much more accurate solutions than the Taylor-series method proposed in [1]. An error analysis for the method is also provided. Applications of the new Taylor-series method to the Volterra integral equation of the second kind [6] and to nonlinear Hammerstein equation [8] were recently made and we reported that the Taylor method provides an excellent means to obtain accurate solutions for these equations.

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