

Wavelets Application to the Petrov-Galerkin Method for Hammerstein Equations

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WAVELETS APPLICATION TO THE PETROV-GALERKIN METHOD FOR HAMMERSTEIN EQUATIONS

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Abstract

The purpose of this paper is two-fold. First, we develop the Petrov-Galerkin method and the iterated Petrov-Galerkin method for a class of nonlinear Hammerstein equations. Alpert [1] established a class of wavelet basis and applied it to approximate solutions of the Fredholm second kind integral equations by the Galerkin method. He then demonstrated an advantage of a wavelet basis application to such equations by showing that the corresponding linear system is sparse. The second purpose of this paper is to study how this advantage of the sparsity can be extended to nonlinear Hammerstein equations.

1 Introduction

In this paper we study the Hammerstein integral equation

$$u(t) - \int_0^1 k(s, t)\psi(s, u(s))ds = f(t), \quad 0 \leq t \leq 1. \quad (1.1)$$

For known functions k , ψ and f , an unknown solution u is to be approximated. We establish in this paper the Petrov-Galerkin and iterated Petrov-Galerkin methods to approximate u . Equation (1.1) arises as a reformulation of two-point boundary value problems with a certain nonlinear boundary condition, [7]. Also multi-dimensional analogues of equation (1.1) appear as various reformulations of an elliptic partial differential equation with nonlinear boundary conditions; see, e.g., [3, 4]. We shall discuss the present methods in relation to one-dimensional Hammerstein equation of (1.1), however, the theory developed in this paper can be extended straightforwardly to multi-dimensional cases.

The Galerkin and collocation methods are two commonly used methods for the numerical solution of the second kind Fredholm equations as well as for the Hammerstein equation (1.1). Many papers have appeared on these methods and a large part of the results presented in these papers were recently compiled in the monograph by Atkinson, [2]. The superconvergence of the iterated Galerkin method and that of the iterated collocation method for Hammerstein equation (1.1) were obtained recently in [9] and in [8] respectively. On the other hand, the

Petrov-Galerkin and iterated Petrov-Galerkin methods are established in [5] for the second kind Fredholm equation

$$u(t) - \int_0^1 k(s, t)u(s)ds = f(t), \quad 0 \leq t \leq 1. \quad (1.2)$$

It is noted in [5] that the Petrov-Galerkin method includes the Galerkin, collocation, and least squares methods as well as many other nonstandard but useful methods as special cases. Unlike the standard Galerkin method, the Petrov-Galerkin method allows the trial space sequence $X_n \subset X$ and the test space sequence $Y_n \subset X^*$ (the dual of X) to be different. This feature provides a greater freedom in choosing a pair of space sequences $\{X_n, Y_n\}$ so as to improve the computational efficiency over the standard Galerkin method while preserving its convergence order.

In [6], Chen, Micchelli and Xu established the Petrov-Galerkin method for equation (1.2) using a class of orthogonal multiwavelets on $[0, 1]$. Results of this paper show that the wavelet Petrov-Galerkin method for second kind Fredholm integral equations yields linear systems having numerically sparse coefficient matrices whose condition number is bounded. They also developed a truncation strategy that leads to linear systems having sparse coefficient matrices, thus providing a framework for fast algorithms for obtaining approximate solutions. Moreover, their approximate solutions have almost optimal order of convergence. In this paper, we use the wavelet basis constructed by Alpert in [1] to examine closely how the advantage of sparsity obtained by the application of the wavelet basis for the linear equations would manifest itself in the nonlinear equations (1.1).

Throughout this paper we will make the following assumptions on ψ , k and f .

1. $\lim_{t \rightarrow \tau} \|k_t - k_\tau\|_\infty = 0, \quad \tau \in [0, 1] \quad (k_a(b) \equiv k(a, b));$
2. $M_1 \equiv \sup_{0 \leq t \leq 1} \int_0^1 |k(t, s)|ds < \infty;$
3. $f \in C[0, 1];$
4. $\psi(s, x)$ is continuous on $[0, 1] \times R$ and there exists $C_1 > 0$ such that $|\psi(s, x_1) - \psi(s, x_2)| \leq C_1|x_1 - x_2|$ for all $x_1, x_2 \in R$;
5. There is a constant C_2 such that the partial derivative $\psi^{(0,1)}$ of ψ with respect to the second variable satisfies $|\psi^{(0,1)}(t, x_1) - \psi^{(0,1)}(t, x_2)| \leq C_2|x_1 - x_2|$ for all $x_1, x_2 \in R$;
6. For $x \in C[0, 1], \quad \psi(\cdot, x(\cdot)), \psi^{(0,1)}(\cdot, x(\cdot)) \in C[0, 1].$

Although 2 follows from 1 by compactness of $[0, 1]$ and continuity, we list 2 explicitly for convenience.

Our first objective in this paper is to establish the Petrov-Galerkin and iterated Petrov-Galerkin methods for the Hammerstein equation (1.1). We will use the theoretical framework of Chen and Xu [5], and a detailed analysis similar to that of Kaneko and Xu [9], and to Vainikko [11]. Our main theorems in this regard are Theorems 2.6 and 2.7.

Our second objective is to establish a framework for fast algorithms for obtaining approximate solutions. As was stated earlier, a utilization of wavelet basis in approximating a solution of the second kind Fredholm equation results in a matrix system that is sparse, see [1] and [6]. The sparsity is obtained from two important characteristics of the wavelet bases; the property of vanishing moments and the property that each wavelet basis element away from the diagonal has a small support. In section 3, we discuss the Petrov-Galerkin method with wavelet bases and its related topics. The main theorem is Theorem 3.4. In section 4, we address the problem of obtaining sparsity results for the Hammerstein equations (1.1). The substitution $z = \psi(\cdot, u(\cdot))$ used in [10] will allow us to accomplish this in the setting of nonlinear equations. With this substitution, it will be shown that the sparsity takes place in the Jacobian matrix during the application of the Newton's iterative method.

2 The Petrov-Galerkin Method for Hammerstein Equation

We begin this section with a brief review of the Petrov-Galerkin method. We follow closely the paper by Chen and Xu, [5]. Let X be a Banach space and X^* its dual space of continuous linear functionals. For each positive integer n , we assume that $X_n \subset X$, $Y_n \subset X^*$, and X_n and Y_n are finite dimensional vector spaces with

$$\dim X_n = \dim Y_n.$$

We further assume the following approximation property.

(H) If $x \in X$ and $y \in Y$, then there are sequences $\{x_n\}$, $\{y_n\}$ with $x_n \in X_n$, $y_n \in Y_n$ for all n such that $x_n \rightarrow x$, and $y_n \rightarrow y$.

Define K by

$$Ku(t) = \int_0^1 k(s, t)u(s)ds$$

and Ψ by

$$\Psi x(t) = \psi(t, x(t)).$$

With the assumption $K, \Psi : X \rightarrow X$, equation (1.1) can be written as

$$u - K\Psi u = f, \quad f \in X. \quad (2.1)$$

Define, for $x \in X$, an element $P_n x \in X_n$ called the *generalized best approximation* from X_n to x with respect to Y_n by the equation

$$\langle x - P_n x, y_n \rangle = 0 \quad \text{for all } y_n \in Y_n. \quad (2.2)$$

It is proved in [5], [Proposition 2.1], that for each $x \in X$, the generalized best approximation from X_n to x with respect to Y_n exists uniquely if and only if

$$Y_n \cap X_n^\perp = \{0\}, \quad (2.3)$$

where $X_n^\perp = \{x^* \in X^* : \langle x, x^* \rangle = 0 \text{ for all } x \in X_n\}$. When condition (2.3) is satisfied, P_n defines a projection; $P_n^2 = P_n$. Throughout the remainder of this paper, condition (2.3) is assumed.

In order to formulate the Petrov-Galerkin method as part of the general projection scheme and render an appropriate error analysis accordingly, it is important to establish that P_n converges pointwise to the identity operator I . To this end, the notion of regular pair is introduced. Assume that, for each n , there is a linear operator $\Pi_n : X_n \rightarrow Y_n$ with $\Pi_n X_n = Y_n$ and satisfying the following two conditions.

$$\textbf{(H-1)} \quad \|x_n\| \leq C_1 \langle x_n, \Pi_n x_n \rangle^{1/2} \text{ for all } x_n \in X_n.$$

$$\textbf{(H-2)} \quad \|\Pi_n x_n\| \leq C_2 \|x_n\| \text{ for all } x_n \in X_n.$$

Here C_1 and C_2 are constants independent of n .

If a pair of space sequences $\{X_n\}$ and $\{Y_n\}$ satisfy (H-1) and (H-2), we call $\{X_n, Y_n\}$ a *regular pair*. It is shown in [5], [Proposition 2.4], that, if a regular pair $\{X_n, Y_n\}$ satisfies $\dim X_n = \dim Y_n$ and condition (H), then the corresponding generalized projection P_n satisfies:

(P1) $\|P_n x - x\| \rightarrow 0$ as $n \rightarrow \infty$ for all $x \in X$,

(P2) $\|P_n\| \leq C$, $n = 1, 2, 3, \dots$, for some constant C , and

(P3) $\|P_n x - x\| \leq c \|Q_n x - x\|$, $n = 1, 2, 3, \dots$, for some constant c ,

where $Q_n x$ is the best approximation from X_n to x .

The Petrov-Galerkin approximation to the Fredholm second kind equation (1.2) is obtained by solving for u_n of the following equation:

$$\langle u_n - Ku_n, y_n \rangle = \langle f, y_n \rangle \quad \text{for all } y_n \in Y_n. \quad (2.4)$$

This is consistent with the basic principle that underlies the projection method, which requires the residual $r_n \equiv u_n - Ku_n - f$ to disappear under a projection. Using the generalized projection P_n , equation (2.4) can be written in operator form as

$$u_n - P_n K u_n = P_n f. \quad (2.5)$$

Therefore, the Petrov-Galerkin method is a projection method.

We recall the following three theorems that will become instrumental in establishing our main theorems in this section.

Theorem 2.1 (see, e.g., [2]) *Let X be a Banach space and $\{X_n\}$ a sequence of finite-dimensional subspaces of X . Assume that $K: X \rightarrow X$ is a compact linear operator not having 1 as an eigenvalue. Let $\hat{P}_n: X \rightarrow X_n$ be a sequence of linear projections that converges pointwise to the identity operator I in X . Then, for sufficiently large n , the equation*

$$u_n - \hat{P}_n K u_n = \hat{P}_n f \quad (2.6)$$

has a unique solution $u_n \in X_n$ with

$$\|u_n - u\| \leq C \|u - \hat{P}_n u\|,$$

where C is a constant independent of n and u is the unique solution of equation (1.1).

Theorem 2.2 ([5]) *Let X be a Banach space and $K: X \rightarrow X$ a compact linear operator. Assume that 1 is not an eigenvalue of the operator K . Suppose that $\{X_n, Y_n\}$ is a regular pair that satisfies $\dim X_n = \dim Y_n$ and condition (H). Then, for any given $f \in X$, there exists a positive integer*

N such that, for all $n \geq N$, the Petrov-Galerkin equation (2.6) has a unique solution $u_n \in X_n$ that satisfies

$$\|u - u_n\| \leq C \inf_{x_n \in X_n} \|u - x_n\|, \quad n \geq N,$$

where C is a constant independent of n and $u \in X$ is the unique solution of equation (1.1).

We will establish an analogous result to Theorem 2.2 for the Hammerstein equation (2.1). To this end, we recall the third theorem below which is due to Vainikko [11]. This useful theorem describes the conditions that are necessary to approximate a solution of a class of nonlinear operator equation.

Theorem 2.3 *Let T and $\tilde{T}$ be continuous over an open set Ω in a Banach space X . Let the equation*

$$x = \tilde{T}x$$

have an isolated solution $\tilde{x}_0 \in \Omega$ and let the following conditions be satisfied:

- (a) *The operator T is Fréchet differentiable in some neighborhood of the point $\tilde{x}_0$ while the linear operator $I - T'(\tilde{x}_0)$ is continuously invertible;*
- (b) *Suppose that for some $\delta > 0$, and $0 < q < 1$, the following inequalities are valid (the number δ is assumed to be so small that the sphere $\|x - \tilde{x}_0\| \leq \delta$ is contained within Ω)*

$$\sup_{\|x - \tilde{x}_0\| \leq \delta} \|(I - T'(\tilde{x}_0))^{-1}(T'(x) - T'(\tilde{x}_0))\| \leq q, \quad (2.7)$$

$$\alpha \equiv \|(I - T'(\tilde{x}_0))^{-1}(T(\tilde{x}_0) - \tilde{T}(\tilde{x}_0))\| \leq \delta(1 - q). \quad (2.8)$$

Then the equation $x = Tx$ has in the sphere $\|x - \tilde{x}_0\| \leq \delta$ a unique solution x_0 . Moreover, the inequality

$$\frac{\alpha}{1 + q} \leq \|x_0 - \tilde{x}_0\| \leq \frac{\alpha}{1 - q} \quad (2.9)$$

is valid.

The Petrov-Galerkin method for the Hammerstein equation (2.1) is defined by

$$\langle u_n - K\Psi u_n, y_n \rangle = \langle f, y_n \rangle \quad \text{for all } y_n \in Y_n. \quad (2.10)$$

Equation (2.10) is equivalent, in operator form, to

$$u_n - P_n K \Psi u_n = P_n f. \quad (2.11)$$

Let

$$\tilde{T}x \equiv K \Psi x + f \quad (2.12)$$

and

$$T_n x_n \equiv P_n K \Psi x_n + P_n f, \quad (2.13)$$

Then equations (2.1) and (2.11) can be written as

$$u = \tilde{T}u \quad (2.14)$$

and

$$u_n = T_n u_n \quad (2.15)$$

respectively. We are now ready to state and prove our first theorem.

Theorem 2.4 *Let u_0 be an isolated solution of equation (2.1) in the Banach space $X = L^\infty[0, 1]$. Assume that 1 is not an eigenvalue of $K \Psi'(u_0)$, where $\Psi'(x_0)$ denotes the Fréchet derivative of Ψ at x_0 . Then the Petrov-Galerkin approximation equation (2.11) has, for each sufficiently large n , a unique solution u_n in some ball of radius δ centered at x_0 , $B(x_0, \delta)$. Further, there exists $0 < q < 1$, independent of n , such that if*

$$\alpha_n \equiv \| (I - T'_n(u_0))^{-1} (T_n(u_0) - \tilde{T}(u_0)) \|$$

then

$$\frac{\alpha_n}{1+q} \leq \| u_n - u \|_\infty \leq \frac{\alpha_n}{1-q}. \quad (2.16)$$

Finally, if the distance from u_0 to X_n is $E_n(u_0) \equiv \inf_{x \in X_n} \| u_0 - x \|_\infty$, then there exists a constant C , independent of n , such that

$$E_n(u_0) \leq \| u_n - u_0 \|_\infty \leq C \{ E_n(K \Psi(u_0)) + E_n(f) \}. \quad (2.17)$$

Proof: We will apply Theorem 2.3 of Vainikko. It requires that T_n be Fréchet differentiable in some neighborhood of u_0 and that $I - T'_n(u_0)$ be continuously invertible. But these follow from assumption 5 of Section 1 and from (P1)-(P3) above. Hence, condition (a) of Theorem 2.3 is

verified. We next check condition (b) of Theorem 2.3. To establish (2.7), we need the uniform boundedness of $(I - T'_n(u_0))^{-1}$ for all sufficiently large n and the local Lipschitz condition

$$\| T'_n(u_0) - T'_n(u) \|_\infty \leq M \| u_0 - u \|_\infty \quad (2.18)$$

for all u in some neighborhood of u_0 . (Here and throughout the remainder of the paper we use M to denote a generic constant the exact value of which may change at each occurrence.) Then, choosing δ sufficiently small, (2.7) holds. But (2.18) holds because for each $\| u \|_\infty \leq 1$,

$$\begin{aligned} \| T'_n(u_0)[x] - T'_n(u)[x] \|_\infty &= \| P_n K \Psi'(u_0)x - P_n K \Psi'(u)x \|_\infty \\ &\leq C \sup_{0 \leq t \leq 1} \left| \int_0^1 k(t, s) [\psi^{(0,1)}(s, u_0(s)) - \psi^{(0,1)}(s, u(s))] x(s) ds \right| \\ &\leq C C_2 M_1 \sup_{0 \leq t \leq 1} |u_0(t) - u(t)| = M \| u_0 - u \|_\infty . \end{aligned}$$

where C and C_2 are constants defined earlier in proposition 2.4 of [5] and in assumption 5 of Section 1 respectively.

The fact that, for sufficiently large n , $I - T'_n(u_0)$ is invertible and $\| (I - T'_n(u_0))^{-1} \|$ is uniformly bounded can be established by observing that for each $x \in X$,

$$\| T'_n(u_0)[x] - T'(u_0)[x] \|_\infty = \| P_n K \Psi'(u_0)[x] - K \Psi'(u_0)[x] \|_\infty \rightarrow 0$$

by Proposition 2.4, [5]. Thus $T'_n(u_0)$ converges in operator norm to $T'(u_0)$, and the result follows from the hypothesis that $(I - T'(u_0))^{-1}$ exists and from the fact that the set of continuously invertible operators is an open set. Hence (2.7) is satisfied.

To check (2.8), we use the previous result;

$$\begin{aligned} \alpha_n &\leq M \| T_n(u_0) - \tilde{T}(u_0) \|_\infty \\ &\leq M \{ \| P_n f - f \|_\infty + \| P_n K \Psi(u_0) - K \Psi(u_0) \|_\infty \} \\ &\rightarrow 0, \text{ as } n \rightarrow \infty. \end{aligned} \quad (2.19)$$

Thus, (2.8) is implied, and (2.16) follows from Theorem 2.3.

For (2.17), the first inequality is trivial, as $u_n \in X_n$. The second inequality follows from (2.16), (2.19) and the fact proved in Proposition 2.4 of [5] that

$$\| P_n K \Psi(u_0) - K \Psi(u_0) \|_\infty \leq M \| Q_n K \Psi(u_0) - K \Psi(u_0) \|_\infty .$$

□

2.1 Iterated Petrov-Galerkin Method

In this subsection, we discuss the iterated Petrov-Galerkin method. This extends the result of the iterated Galerkin method for Hammerstein equation obtained in [9]. Define the iterated approximation for Hammerstein equation by

$$x'_n = f + K\Psi x_n. \quad (2.20)$$

Then as in [10], [9], we find that the iterated approximation x'_n satisfies

$$x'_n - K\Psi P_n x'_n = f. \quad (2.21)$$

Define $\hat{T}_n$ by

$$\hat{T}_n x = K\Psi P_n x + f. \quad (2.22)$$

The following is instrumental to the stability of the iterated approximation whose effectiveness will be demonstrated in Theorem 2.6 below.

Theorem 2.5 *Let $x_0 \in L^\infty[0, 1]$ be an isolated solution of (1.1). Assume 1 is not an eigenvalue of $(K\Psi)'(x_0)$. Then, $I - \hat{T}'_n(x_0)$ is invertible and there exists $L > 0$ such that $\| (I - \hat{T}'_n(x_0))^{-1} \| \leq L$ for all sufficient large n ,*

Proof: By Hölder's inequality and hypotheses 2 and 5,

$$\begin{aligned} \| \tilde{T}'(x_0)[x] - \hat{T}'_n(x_0)[x] \|_\infty &= \| \int_0^1 k(s, \cdot) \psi^{(0,1)}(s, x_0(s)) x(s) - k(s, \cdot) \psi^{(0,1)}(s, P_n x_0(s)) x(s) ds \|_\infty \\ &\leq M_1 \| (\psi^{(0,1)}(\cdot, x_0) - \psi^{(0,1)}(\cdot, P_n x_0)) x \|_\infty \\ &\leq MC_2 \| x_0 - P_n x_0 \|_\infty \| x \|_\infty \rightarrow 0, \quad n \rightarrow \infty. \end{aligned}$$

Thus $\hat{T}'_n(x_0)$ converges to $\tilde{T}'(x_0)$ pointwise.

Note also that

$$\begin{aligned} \| \psi^{(0,1)}(\cdot, P_n x_0) \|_\infty &\leq \| \psi^{(0,1)}(\cdot, P_n x_0) - \psi^{(0,1)}(\cdot, x_0) \|_\infty + \| \psi^{(0,1)}(\cdot, x_0) \|_\infty \\ &\leq C_2 \| P_n x_0 - x_0 \|_\infty + \| \psi^{(0,1)}(\cdot, x_0) \|_\infty \\ &\rightarrow \| \psi^{(0,1)}(\cdot, x_0) \|_\infty \quad n \rightarrow \infty. \end{aligned}$$

Thus $\| \psi^{(0,1)}(\cdot, P_n x_0) \|_\infty$ is bounded uniformly in n , say $\| \psi^{(0,1)}(\cdot, P_n x_0) \|_\infty \leq M_2$. Then

$$\| \hat{T}'_n(x_0)[x] \|_\infty = \| K\Psi'(P_n x_0)[x] \|_\infty \leq M_1 M_2 \| x \|_\infty.$$

This implies that $\hat{T}'_n$ is collectively compact. By [1], $\| (I - \hat{T}'_0(x_0))^{-1} \|_\infty$ exists for all sufficiently large n and there exists $L > 0$ such that $\| (I - \hat{T}'_n(x_0))^{-1} \|_\infty \leq L$ for these n . $\square$

Theorem 2.6 *Let $x_0 \in L^\infty[0, 1]$ be an isolated solution of (1.1). Assume 1 is not an eigenvalue of $(K\Psi)'(x_0)$. Then there exists $\delta > 0$ such that equation (2.21) has, for each sufficiently large n , a unique solution x'_n in $B(x_0, \delta)$. Further if*

$$\beta_n \equiv \| (I - \hat{T}'_n(x_0))^{-1} (\hat{T}_n(x_0) - \tilde{T}(x_0)) \|_\infty,$$

then x'_n satisfies

$$\frac{\beta_n}{1+q} \leq \| x_0 - x'_n \|_\infty \leq \frac{\beta_n}{1-q},$$

for some $0 < q < 1$.

Proof: First notice that hypothesis (a) of Theorem 2.3 holds; for

$$\begin{aligned} \| (I - \hat{T}'_n(x_0))^{-1} [\hat{T}'_n(x) - \hat{T}'_n(x_0)] \|_\infty &\leq M \| \hat{T}'_n(x) - \hat{T}'_n(x_0) \|_\infty \\ &= \| K\Psi'(P_n x) - K\Psi'(P_n x_0) \|_\infty. \end{aligned}$$

But for $y \in C[0, 1]$, we have

$$\begin{aligned} &\| K\Psi'(P_n x)[y] - K\Psi'(P_n x_0)[y] \|_\infty \\ &= \| \int_0^1 k(s, \cdot) [\psi^{(0,1)}(s, P_n x(s)) - \psi^{(0,1)}(s, P_n x_0(s))] y(s) ds \|_\infty \\ &\leq \| \int_0^1 |k(s, \cdot)| C_2 |P_n x(s) - P_n x_0(s)| |y(s)| ds \|_\infty \\ &\leq M_1 C_2 \| P_n x - P_n x_0 \|_\infty \| y \|_\infty. \end{aligned}$$

The two previous results and Theorem 3.4 of [5], concerning the uniform boundedness of the generalized projection P_n , yield a constant C such that

$$\begin{aligned} \| (I - \hat{T}'_n(x_0))^{-1} [\hat{T}'_n(x) - \hat{T}'_n(x_0)] \|_\infty &\leq M_1 C_2 \| P_n x - P_n x_0 \|_\infty \\ &\leq C \| x - x_0 \|_\infty. \end{aligned}$$

Next, by the Lipschitz condition on ψ ,

$$\begin{aligned} \beta_n &\equiv \| (I - \hat{T}'_n(x_0))^{-1} (\hat{T}_n(x_0) - \tilde{T}(x_0)) \|_\infty \\ &\leq M \| K\Psi P_n x_0 - K\Psi x_0 \|_\infty \\ &= M \| \int_0^1 k(s, \cdot) [\psi(s, P_n x_0(s)) - \psi(s, x_0(s))] ds \|_\infty \\ &\leq M_1 M C_1 \| P_n x_0 - x_0 \|_\infty \rightarrow 0 \quad \text{as } n \rightarrow \infty. \end{aligned}$$

The application of Theorem 2.3 completes the proof. $\square$

Our next task is to establish the superconvergence of the iterated Petrov-Galerkin approximation x'_n . Notice that a necessary and sufficient condition for the superconvergence is

$$\beta_n = o(\alpha_n). \quad (2.23)$$

Let us obtain (2.23) by estimating β_n using the method to prove theorem 3.3, [9]. First of all,

$$\begin{aligned} \beta_n &\leq M \| \hat{T}_n(x_0) - \hat{T}(x_0) \|_\infty \\ &= M \| K[\Psi(P_n x_0) - \Psi(x_0)] \|_\infty \\ &= M \| \int_0^1 k(s, \cdot) [\psi(s, P_n x_0(s)) - \psi(s, x_0(s))] ds \|_\infty. \end{aligned} \quad (2.24)$$

Define

$$d(t) \equiv \left| \int_0^1 k(s, t) [\psi(s, P_n x_0(s)) - \psi(s, x_0(s))] ds \right|.$$

By the mean value theorem, there exists $0 < \theta < 1$ such that

$$d(t) = \left| \int_0^1 k(s, t) \psi^{(0,1)}(s, x_0(s) + \theta(P_n x_0(s) - x_0(s))) (x_0(s) - P_n x_0(s)) ds \right|$$

Note that $\int_0^1 u(s) [x_0(s) - P_n x_0(s)] ds = \langle u, x_0 - P_n x_0 \rangle = 0$ for all $u \in Y_n$. Thus, for all $u \in Y_n$,

$$\begin{aligned} d(t) &= \left| \int_0^1 k(s, t) [\psi^{(0,1)}(s, x_0(s) + \theta(P_n x_0(s) - x_0(s))) - u(s)] [x_0(s) - P_n x_0(s)] ds \right| \\ &\leq \left| \int_0^1 k(s, t) \psi^{(0,1)}(s, x_0(s) + \theta(P_n x_0(s) - x_0(s))) [x_0(s) - P_n x_0(s)] \right. \\ &\quad \left. - k(s, t) \psi^{(0,1)}(s, x_0(s)) [x_0(s) - P_n x_0(s)] ds \right| \\ &\quad + \left| \int_0^1 k(s, t) [\psi^{(0,1)}(s, x_0(s)) - u(s)] [(x_0(s) - P_n x_0(s))] ds \right| \\ &\equiv T_1(t) + T_2(t). \end{aligned}$$

Assuming that $x_0 \in L^\infty[0, 1]$, we proceed as follows:

$$\begin{aligned} T_1(t) &\leq C_1 \int_0^1 |k(s, t)| \theta |P_n x_0(s) - x_0(s)|^2 ds \\ &\leq C \| P_n x_0 - x_0 \|_\infty^2 \int_0^1 |k(s, t)| ds, \end{aligned}$$

So

$$\| T_1 \|_\infty \leq M_1 \| P_n x_0 - x_0 \|_\infty^2. \quad (2.25)$$

We also have

$$T_2(t) \leq \left[\int_0^1 |k(s, t)| \psi^{(0,1)}(s, x_0(s)) - u(s) |^q ds \right]^{1/q} \left[\int_0^1 |x_0(s) - P_n x_0(s)|^p ds \right]^{1/p}. \quad (2.26)$$

where $\frac{1}{p} + \frac{1}{q} = 1, p, q \geq 1$ with $p = 1$ and $q = \infty$.

Summarizing the discussion above, we arrive at the following theorem establishing the superconvergence of the iterated Petrov-Galerkin approximation for Hammerstein equations. It generalizes theorem 3.3 of [9] by extending the condition $x_0 \in C[0, 1]$ to $x_0 \in L^\infty[0, 1]$.

Theorem 2.7 *Under the hypotheses of Theorem 2.3, there is a constant C such that*

$$\begin{aligned} \|x_0 - x'_n\|_\infty &\leq C\{\|P_n x_0 - x_0\|_\infty^2 \\ &\quad + \int_0^1 |k(s, t)|\psi^{(0,1)}(s, x_0(s)) - u(s)|^q ds\}^{1/q} [\int_0^1 |x_0(s) - P_n x_0(s)|^p ds]^{1/p}. \end{aligned}$$

3 The Wavelet Petrov-Galerkin Method

Alpert [1] constructed a class of wavelet basis in $L^2[0, 1]$ and applied it to approximate the solution of the Fredholm equation (1.2). The numerical method employed in [1] is the Galerkin method. The sparsity of the matrix in the associated linear system is obtained using the vanishing moment property of the wavelets as well as using the notion of ‘separation from the diagonal’ for the supports of the wavelet basis elements. We provide below a brief review of Alpert’s wavelets and how they can be applied to produce this sparsity. The wavelet basis for $L^2[0, 1]$ is comprised of dilates and translates of a set of functions $h_1, h_2, \dots, h_k$. In particular, for k a positive integer, and for $m = 0, 1, \dots$ we define a space S_m^k of piecewise polynomial functions,

$$S_m^k = \{f: \text{the restriction of } f \text{ to the interval } (2^{-m}n, 2^{-m}(n+1)) \text{ is a polynomial of degree less than } k, \text{ for } n = 0, \dots, 2^m - 1, \text{ and } f \text{ vanishes elsewhere}\}. \quad (3.1)$$

Note that $\dim S_m^k = 2^m k$ and

$$S_0^k \subset S_1^k \subset \dots \subset S_m^k \subset \dots$$

The orthogonal complement of S_m^k in S_{m+1}^k is denoted by R_m^k so that $\dim R_m^k = 2^m k$ and

$$S_m^k \oplus R_m^k = S_{m+1}^k, \quad R_m^k \perp S_m^k.$$

Note that

$$S_m^k = S_0^k \oplus R_0^k \oplus R_1^k \oplus \dots \oplus R_{m-1}^k. \quad (3.2)$$

The set of functions $h_1, h_2, \dots, h_k$ mentioned above is taken as an orthonormal basis for R_0^k . Since R_0^k is orthogonal to S_0^k , the first k moments of $h_1, h_2, \dots, h_k$ vanish,

$$\int_0^1 h_j(s) s^i dx = 0, \quad i = 0, 1, \dots, k-1. \quad (3.3)$$

The wavelet basis of Alpert is constructed by defining orthonormal systems

$$h_{j,m}^n(s) = 2^{m/2} h_j(2^m s - n), \quad j = 1, \dots, k; \quad m, n \in \mathbb{Z}. \quad (3.4)$$

We refer the reader to [1] for detailed constructions of $h_1, h_2, \dots, h_k$. The functions $h_{j,m}^n$ generated in (3.4) become an orthonormal basis for R_m^k :

$$R_m^k = \text{linear span } \{h_{j,m}^n: j = 1, \dots, k; n = 0, \dots, 2^m - 1\}.$$

If we let $\{u_1, \dots, u_k\}$ denote an orthonormal basis for S_0^k , then the orthonormal system

$$\begin{aligned} B_k = & \{u_j: j = 1, \dots, k\} \\ & \cup \{h_{j,m}^n: j = 1, \dots, k; m = 0, 1, \dots; n = 0, 1, \dots, 2^m - 1\} \end{aligned}$$

becomes the multiwavelet basis of order k for $L^2[0, 1]$. In practice, we take an arbitrarily large but a fixed value for m and use

$$\begin{aligned} \tilde{B}_k &= \{u_j: j = 1, \dots, k\} \cup \{h_{j,m}^n: j = 1, \dots, k; n = 0, 1, \dots, 2^m - 1\} \\ &\equiv \{b_j\}_{j=1}^{k(2^m+1)} \end{aligned} \quad (3.5)$$

for an orthonormal basis for S_m^k . The approximating power of the wavelets is given as follows [1]:

Lemma 3.1 *Let Q_m^k be the orthogonal projection of $L^2[0, 1]$ onto S_m^k . If $f \in C^k[0, 1]$, then*

$$\|Q_m^k f - f\| \leq 2^{-mk} \frac{2}{4^k k!} \sup_{x \in [0, 1]} |f^{(k)}(x)|.$$

The Galerkin method for approximating the solution of the Fredholm equation (1.2) using the basis b_j in (3.5) requires computations of

$$K_{ij} = \int_0^1 \int_0^1 k(t, s) b_i(t) b_j(s) dt ds. \quad (3.6)$$

What Alpert observed at this point is that a large majority of K_{ij} can be neglected from actual computations because of their insignificant sizes, resulting in a sparse matrix for the linear system for the Galerkin method. More precisely, he defined the following regarding the separation from the diagonal of a support of $b_i \otimes b_j$.

Definition We say that a rectangular region $[t, t + a] \times [s, s + b] \subset \mathbb{R}^2$ is separated from the diagonal if $a + \max\{a, b\} \leq s - t$ or $b + \max\{a, b\} \leq t - s$.

For $k(t, s) = \log |t - s|$, the following are proved in [1].

Lemma 3.2 Suppose that $k(t, s) = \log |t - s|$ is given and B_k is the multiwavelet basis of order k . Denote the supports of $b_i(t)$ and $b_j(s)$ in (3.6) by $[x_0, x_0 + a]$ and $I_j = [y_0, y_0 + b]$ respectively and assume that they are separated from the diagonal. Then

$$|K_{ij}| \leq \frac{\sqrt{ab}}{2^k \cdot 3^{k-1}}.$$

Lemma 3.3 Suppose that $I_1, I_2, \dots, I_n$ are the nonincreasing intervals of support of the first n functions of the basis b_k . Of the n^2 rectangular regions $I_i \times I_j$, we denote the number separated from the diagonal by $S(n)$ and the number “near” the diagonal by $N(n) = n^2 - S(n)$. Then $N(n) = O(n \log n)$.

Equation (2.11), when approximated by the Galerkin method with the wavelet basis, requires computations of the following integrals. Here the approximate solution is written as $u_n(s) = \sum_{i=1}^n a_i b_i(s)$ for some $a_i \in R$.

$$\int_0^1 \int_0^1 k(t, s) \psi(x, \sum_{i=1}^n a_i b_i(s)) b_j(t) ds dt. \quad (3.7)$$

The unknowns a_i are, in many applications, found by an iterative scheme. Note that, at each step of iteration, the integrals in (3.7) must be recomputed as each a_i has different values, making the computations costly. To circumvent the difficulty, we use the technique that was developed in [10]. If we let

$$z = \Psi u,$$

in (2.11), then

$$u = Kz + f.$$

Hence z satisfies

$$z = \Psi(Kz + f). \quad (3.8)$$

We approximate this equation by

$$z_n(t) = \psi(t, \int_0^1 z_n(s) k(t, s) ds + f(t)),$$

where $z_n(t) = \sum_{i=1}^n a_i b_i(t) \in X_n$ is computed via

$$\langle z_n, y \rangle = \langle \Psi(Kz_n + f), y \rangle, \quad y \in Y_n. \quad (3.9)$$

We note that the Petrov-Galerkin method used in (3.9) is distinct from the collocation method of Kumar and Sloan in [10] to solve equation (3.8). Equation (3.9) is equivalent to (compare this with (3.7))

$$\sum_{i=1}^n a_i \int_0^1 b_i(t) b_j^*(t) dt = \int_0^1 \psi(t, \sum_{i=1}^n a_i \int_0^1 k(t, s) b_i(s) ds + f(t)) b_j^*(t) dt, \quad j = 1, \dots, n. \quad (3.9')$$

where $\{b_j^*\}$ is a wavelet basis for Y_n . Note that the unknown a_i 's now appear outside the third integral, allowing us to compute the integral $\int_0^1 k(t, s) b_i(s) ds$ for each i only once throughout the iterative process for approximating a_i 's. It is important to maintain the condition $\dim X_n = \dim Y_n$. In operator notation, equation (3.9) becomes

$$z_n = P_n \Psi(K z_n + f), \quad (3.10)$$

where P_n is the generalized projection onto Y_n . Define T_n by

$$T_n z \equiv P_n \Psi(K z + f).$$

Then (3.10) can be written as

$$z_n = T_n z_n.$$

Let $\tilde{T}$ be defined as

$$\tilde{T} z \equiv \Psi(K z + f),$$

so that equation (3.8) becomes

$$z = \tilde{T} z.$$

Theorem 3.4 *Let $z_0 \in L^\infty[0, 1]$ be an isolated solution of (3.8) with $f \in L^\infty[0, 1]$. Assume that 1 is not an eigenvalue of $\Psi'(K z_0 + f)K$ and that $\psi^{(0,1)}(t, (K z)(t) + f(t)) \in L^\infty[0, 1]$ for each $z \in L^\infty[0, 1]$. If $K: L^\infty[0, 1] \rightarrow L^\infty[0, 1]$, then there exists $\delta > 0$ such that equation (3.9) has a unique solution $z_n \in B(z_0, \delta)$ for sufficiently large n . Further if*

$$\alpha_n \equiv \| (I - T_n'(z_0))^{-1} (T_n z_0 - \tilde{T} z_0) \|,$$

then z_n satisfies

$$\frac{\alpha_n}{1+q} \leq \| z_0 - z_n \| \leq \frac{\alpha_n}{1-q},$$

for some $0 < q < 1$, and

$$\| z_0 - z_n \| \leq C 2^{-mk} \frac{2}{4^k k!} \sup_{x \in [0,1]} |g^{(k)}(x)|.$$

provided that $g \equiv \Psi(K z_0 + f) \in C^k[0, 1]$.

Proof: Since $K : X \rightarrow X$ and for $z \in X$, $\psi^{(0,1)}(t, (Kz)(t) + f(t)) \in X$, the Fréchet derivative T'_n exists in a neighborhood of z_0 and

$$T'_n(z_0)[z](t) = P_n \psi^{(0,1)}(t, \int_0^1 k(t, s) z_0(s) ds + f(t)) \int_0^1 k(t, s) z(s) ds.$$

Since P_n converges pointwise to the identity, $T'_n(z_0)(z)$ converges to $\Psi'(Kz_0 + f)(K(z))$, where $\Psi'(x_0)(x)(t) = \psi^{(0,1)}(t, x_0(t))x(t)$. As 1 is not an eigenvalue of $\Psi'(Kz_0 + f)K$, we find that $I - T'_n(z_0)$ is continuously invertible for all sufficiently large n . Thus hypothesis (a) of Theorem 2.3 holds.

Suppose $\| (I - T'_n(z_0))^{-1} \| \leq L$ for some $L > 0$. We then have

$$\| (I - T'_n(z_0))^{-1} (T'_n(z) - T'_n(z_0)) \| \leq L \| T'_n(z) - T'_n(z_0) \|,$$

and for $\| w \| \leq 1$, using a Lipschitz condition on Ψ' , we obtain

$$\begin{aligned} \| T'_n(z)(w) - T'_n(z_0)(w) \| &= \| P_n \Psi'(Kz + f)(K(w)) - P_n \Psi'(Kz_0 + f)(K(w)) \| \\ &\leq M \| (\Psi'(Kz + f) - \Psi'(Kz_0 + f))(K(w)) \| \\ &\leq M \| \Psi'(Kz + f) - \Psi'(Kz_0 + f) \| \| w \| \\ &\leq M \| z - z_0 \|. \end{aligned}$$

Hence condition (2.7) in hypothesis (b) of theorem 2.3 is satisfied for δ sufficiently small (with any q , $0 < q < 1$).

Finally, since $\| (I - T'_n(z_0))^{-1} \| \leq L$ and

$$\| \tilde{T}(z_0) - T_n(z_0) \| = \| \Psi(Kz_0 + f) - P_n \Psi(Kz_0 + f) \|$$

converges to zero (by pointwise convergence of P_n to the identity), condition (2.8) in hypothesis (b) of theorem 2.3 is satisfied.

The application of Theorem 2.3 yields

$$\frac{\alpha_n}{1+q} \leq \| z_0 - z_n \| \leq \frac{\alpha_n}{1-q},$$

Finally, lemma 3.1 and (P3) give

$$\| z_0 - z_n \| \leq C 2^{-mk} \frac{2}{4^k k!} \sup_{t \in [0,1]} |g^{(k)}(t)|,$$

since $\alpha_n \leq C \| (I - P_n)(\Psi(Kz_0 + f)) \|$. $\square$

We now observe the superconvergence of u_n to u in L^∞ norm. Since $u_0 = Kz_0 + f$ and $u_n = Kz_n + f$, for any $\phi_n \in S_m^k$,

$$\begin{aligned}
\|u_0 - u_n\|_\infty &= \|K(z_0 - z_n)\|_\infty \\
&= \sup_{t \in [0,1]} \left| \int_0^1 k(t, s)(z_0(s) - z_n(s))ds \right| \\
&= \sup_{t \in [0,1]} | \langle k_t, z_0 - z_n \rangle | \\
&= \sup_{t \in [0,1]} | \langle k_t - \phi_n, z_0 - z_n \rangle | \\
&\leq \sup_{t \in [0,1]} \|k_t - \phi_n\| \|z_0 - z_n\|.
\end{aligned} \tag{3.11}$$

Let $D^{(k)}u$ denote the generalized derivative of u of order k . Define the Sobolev space of functions as follows:

$$W_2^k = \{u : D^{(i)}u \in L^2(0, 1), \text{ for } i = 0, 1, \dots, k\}.$$

The space is endowed with the norm

$$\|u\|_{W_2^k} = \sum_{i=0}^k \|D^{(i)}u\|.$$

From (3.11) and lemma 3.1, we have the following:

Theorem 3.5 *We assume that $\sup_{t \in [0,1]} \|k_t\|_{W_2^k} < \infty$ and that all the hypotheses of Theorem 3.4 hold. Then*

$$\|u_0 - u_n\|_\infty \leq C(k) 2^{2(1-mk)},$$

where $C(k)$ is a constant that depends upon k but not upon m .

3.1 The Iterated Solution z'_n

Equation (3.10) suggests that we investigate the iterated variant of z_n . Namely, we define

$$z'_n \equiv \Psi(Kz_n + f). \tag{3.12}$$

and

$$u'_n \equiv Kz'_n + f. \tag{3.13}$$

From (3.10) and (3.12),

$$P_n z'_n = P_n \Psi(Kz_n + f) = z_n. \tag{3.14}$$

Equation (3.12) subsequently gives the following equation for z'_n ,

$$z'_n = \Psi(KP_n z'_n + f). \quad (3.15)$$

The next two theorems provide the rates of convergence of z'_n and u'_n to z_0 . The proofs are only outlined as they are similar to the one given for theorem 3.4. Here $\tilde{T}z = \Psi(Kz + f)$ as defined earlier and

$$\hat{T}_n z \equiv \Psi(KP_n z + f).$$

Theorem 3.6 *Let $z_0 \in L^\infty[0, 1]$ be an isolated solution of (3.8) with $f \in L^\infty[0, 1]$. Assume that 1 is not an eigenvalue of $\Psi'(KP_n z_0 + f)KP_n$ and that $\psi^{(0,1)}(t, (KP_n z)(t) + f(t)) \in L^\infty[0, 1]$ for each $z \in L^\infty[0, 1]$. If $K: L^\infty[0, 1] \rightarrow L^\infty[0, 1]$, then there exists $\delta > 0$ such that equation (3.14) has a unique solution $z'_n \in B(z_0, \delta)$ for sufficiently large n . Further if*

$$\beta_n \equiv \| (I - \hat{T}'_n(z_0))^{-1}(\hat{T}_n z_0 - \tilde{T}z_0) \|,$$

then z'_n satisfies

$$\frac{\beta_n}{1+q} \leq \| z_0 - z'_n \| \leq \frac{\beta_n}{1-q}, \quad (3.16)$$

for some $0 < q < 1$, and

$$\| z_0 - z'_n \| \leq C 2^{-mk} \frac{2}{4^k k!} \sup_{t \in [0,1]} |z_0^{(k)}(t)|. \quad (3.17)$$

provided that $z_0 \in C^k[0, 1]$. Finally, if $\sup_t \in [0, 1] \|z_0\|_{W_t^2} < \infty$, then

$$\| z_0 - z'_n \|_\infty \leq C(k) 2^{2(1-mk)}, \quad (3.18)$$

where $C(k)$ depends only upon k .

Proof: The Fréchet derivative $\hat{T}'_n$ exists in a neighborhood of z_0 and

$$\begin{aligned} \hat{T}'_n(z_0)[z](t) &= \Psi'(KP_n z_0 + f)(KP_n z)(t) \\ &= \psi^{(0,1)}(t, \int_0^1 k(t, s) P_n z_0(s) ds + f(t)) \int_0^1 k(t, s) P_n z(s) ds. \end{aligned}$$

Since 1 is not an eigenvalue of $\hat{T}'_n(z_0)$, $(I - \hat{T}'_n(z_0))^{-1}$ exists and uniformly bounded for sufficiently large n . We also note that for $\|w\| < 1$,

$$\|\hat{T}'_n(z)(w) - \hat{T}'_n(z_0)(w)\| \leq M \|z - z_0\|.$$

Moreover,

$$\|\tilde{T}(z_0) - \hat{T}_n(z_0)\| = \|\Psi(Kz_0 + f) - \Psi(KP_n z_0 + f)\| \leq M\|z_0 - P_n z_0\|.$$

Applications of theorem 2.3 and lemma 3.1 yield (3.16) and (3.17). To obtain (3.18),

$$\begin{aligned} \|z_0 - z'_n\|_\infty &\leq M\|\tilde{T}(z_0) - \hat{T}_n(z_0)\|_\infty \\ &= \|\Psi(Kz_0 + f) - \Psi(KP_n z_0 + f)\|_\infty \\ &\leq M\|K(z_0 - P_n z_0)\|_\infty \\ &= M \sup_{t \in [0,1]} |< k_t, z_0 - P_n z_0 >|, \end{aligned}$$

and follow the argument that was used to establish theorem 3.5. $\square$

Theorem 3.7 *We assume that $\sup_{t \in [0,1]} \|k_t\|_{W_2^k} < \infty$ and that all the hypotheses of Theorem 3.6 hold. Then*

$$\|u_0 - u'_n\|_\infty \leq C(k) 2^{4(1-mk)},$$

where $C(k)$ is a constant that depends upon k but not upon m .

Proof: From (3.13), we obtain

$$\|u_0 - u'_n\|_\infty = \|Kz_0 + f - K(z'_n + f)\|_\infty = \|K(z_0 - z'_n)\|_\infty.$$

$\square$

4 Sparsity in Nonlinear Equations

In this section, we examine closely some issues associated with solving equations in (3.9'). Particularly, we propose to approximate $\{a_i\}$ by the Newton's method. Before we proceed, we recall from (3.2) that $\dim S_m^k = 2^m k$. With $n \equiv 2^m k$ for some nonnegative integers m and k , we let $X_n = S_m^k$. The main advantage of the Petrov-Galerkin method is the flexibility for choosing Y_n in the regular pair different from X_n so as to simplify the computations in (3.9'). In the current situation, this can be attained by taking a lower order wavelet to form Y_n . Let $k' < k$ and $Y_n \equiv S_{m'}^{k'}$. The condition $\dim X_n = \dim Y_n$ demands $2^m k = 2^{m'} k'$. For convenience, we take k and k' to satisfy $\frac{k}{k'} = 2^q$ for some nonnegative integer q . If $q = 0$, we have the traditional Galerkin method. Note that $m' = m + q$. As in section 3, we let $X_n = \text{span}\{b_i\}$ and $Y_n = \text{span}\{b_j^*\}$. The following lemma generalizes Lemma 3.3.

Lemma 4.1 Suppose that $I_1, I_2, \dots, I_n$ and $I_1^*, I_2^*, \dots, I_n^*$ are the nonincreasing intervals of support of the bases elements b_i and b_j^* respectively. Of the n^2 rectangular regions $I_i \times I_j^*$, we denote the number separated from the diagonal by $S(n)$ and the number “near” the diagonal by $N(n) = n^2 - S(n)$. Then $N(n) = \mathcal{O}(n \log n)$. In particular, for $n = 2^m k = 2^{m'} k'$, we have

$$N(n) \leq 6mkn - (6 - 2^{3-m})kn - 9k'n + (8 + 6m)kk'.$$

Proof: We define $S_1(p)$ to be the number of pairs (i, j) such that the rectangular region $I_i \times I_j^*$ is separated from the diagonal and $m(I_i) = m(I_j^*) = 2^{-p}$ where m denotes the Lebesgue measure. We have $S_1(p) = (2^p - 1)(2^p - 2)kk'$ for $p = 0, 1, \dots$. We further define $S_2(p, q)$ to be the number of pairs (i, j) such that $I_i \times I_j^*$ is separated from the diagonal and $m(I_i) = 2^{-p}$ and $m(I_j^*) = 2^{-q}$. As in [1], $S_2(p, q) = S_1(\min\{p, q\})2^{|p-q|}$ for $p, q = 0, 1, 2, \dots$. Then

$$\begin{aligned} S(n) &= \sum_{p=0}^{m-1} (S_1(p) + \sum_{q=p+1}^{m'-1} (S_2(p, q) + S_2(q, p))) \\ &= \sum_{p=0}^{m-1} S_1(p) [1 + 2 \sum_{q=p+1}^{m'-1} 2^{q-p}] \\ &= \sum_{p=0}^{m-1} (2^p - 1)(2^p - 2)kk' [1 + 4(2^{m'-1-p} - 1)] \\ &\geq [2 \cdot 2^{m'+m} - 2 \cdot 2^{m'} - 6m2^{m'} - 2^{m'-m+3} + 8 \cdot 2^{m'} \\ &\quad - 2^{m'+m} - 8 + 9 \cdot 2^m - 6m]kk' \\ &= n^2 - 6mkn + (6 - 2^{3-m})kn + 9k'n - (8 + 6m)kk'. \end{aligned}$$

This proves $N(n) = \mathcal{O}(n \log n)$. $\square$

Now define

$$\begin{aligned} f_j(a_1, a_2, \dots, a_n) &\equiv \sum_{i=1}^n a_i \int_0^1 b_i(t) b_j^*(t) dt \\ &\quad - \int_0^1 \psi(t, \sum_{i=1}^n a_i \int_0^1 k(t, s) b_i(s) ds + f(t)) b_j^*(t) dt, \quad j = 1, \dots, n, \end{aligned} \quad (4.1)$$

so that

$$\frac{\partial f_j}{\partial a_l} = \int_0^1 b_l(t) b_j^*(t) dt - \int_0^1 \psi^{(0,1)}(t, \sum_{i=1}^n a_i \int_0^1 k(t, s) b_i(s) ds + f(t)) \int_0^1 k(t, s) b_l(s) b_j^*(t) dt ds, \quad (4.2)$$

for $1 \leq j, l \leq n$.

Let $\bar{a} \equiv (a_1, a_2, \dots, a_n)^T$ and $F(\bar{a}) \equiv (f_1(\bar{a}), f_2(\bar{a}), \dots, f_n(\bar{a}))^T$. Denote the Jacobian matrix for F at $\bar{a}$ by $J(\bar{a})$, $J(\bar{a}) = (\frac{\partial f_j}{\partial a_k})_{j,k=1}^n$. Using the Newton method, with $\bar{a}_0 \in R^n$ given,

$$\bar{a}_{\alpha+1} = \bar{a}_\alpha - J(\bar{a}_\alpha)^{-1} F(\bar{a}_\alpha), \quad \text{for each } \alpha = 0, 1, \dots \quad (4.3)$$

Let us examine the terms in (4.2). If $I_l \times I_j^*$ is separated from the diagonal, $m(I_l \cap I_j^*) = 0$ and

$$\int_0^1 b_l(t) b_j^*(t) dt = \int_{I_l \cap I_j^*} b_l(t) b_j^*(t) dt = 0.$$

Lemma 4.1 shows thus that only $\mathcal{O}(n \log n)$ of such integrals need computation. Now, consider the second term on the right side of (4.2). If we put $\Psi(t) \equiv \psi^{(0,1)}(t, \sum_{i=1}^n a_i \int_0^1 k(t, s) b_i(s) ds + f(t))$, then by assumption 6, there is a constant $M > 0$ such that $\sup_{t \in [0,1]} |\Psi(t)| \leq M$, and

$$\begin{aligned} & \left| \int_0^1 \psi^{(0,1)}(t, \sum_{i=1}^n a_i \int_0^1 k(t, s) b_i(s) ds + f(t)) \int_0^1 k(t, s) b_l(s) b_j^*(t) dt ds \right| \\ &= \left| \int_0^1 \int_0^1 \Psi(t) k(t, s) b_l(s) b_j^*(t) ds dt \right| \\ &\leq M \left| \int_0^1 \int_0^1 k(t, s) b_l(s) b_j^*(t) ds dt \right|. \end{aligned}$$

Even though b_l and b_j^* are elements of wavelet bases of different orders, a result similar to Lemma 3.2 can be obtained in the present setting. For $k(t, s) = \log |t - s|$ and $K_{l,j}^* = \int_0^1 \int_0^1 k(t, s) b_l(s) b_j^*(t) ds dt$, the following theorem describes the size of $K_{l,j}^*$. Despite the differences in the orders of wavelets, an argument can be made so that an upper bound for $K_{l,j}^*$ contains only the parameter k of higher order wavelet. This is because the vanishing moment property of the wavelet basis element is utilized only relative to b_l whose order is k . The wavelet basis element b_j^* contributes in controlling the size of $K_{l,j}^*$ by virtue of the size of its support. The proof is given here for completeness.

Lemma 4.2 *Let B_k and $B_{k'}^*$ be the multiwavelet bases of orders k and k' respectively, where $2^m k = 2^{m'} k'$ and $\frac{k}{k'} = 2^q$ for some nonnegative integer q . Denote the supports I_l of $b_l \in B_k$ and I_j^* of $b_j^* \in B_{k'}^*$ by $[x_0, x_0 + a]$ and $[y_0, y_0 + b]$ respectively and assume that they are separated from the diagonal. Then*

$$|K_{l,j}^*| \leq \frac{\sqrt{ab}}{2k \cdot 3^{k-1}}.$$

Proof:

$$\begin{aligned}
|K_{l,j}^*| &= \left| \int_{y_0}^{y_0+b} \int_{x_0}^{x_0+a} k(t,s) b_l(s) b_j^*(t) ds dt \right| \\
&\leq \int_{y_0}^{y_0+b} \left| \int_{x_0}^{x_0+a} [\log(x_0 + \frac{a}{2} - t) \right. \\
&\quad \left. - \sum_{m=1}^{\infty} \frac{1}{m} \left(\frac{x_0+a/2-t}{x_0+a/2-t} \right)^m] b_l(s) ds \right| |b_j^*(t)| dt \\
&\leq \int_{y_0}^{y_0+b} \left| \int_{x_0}^{x_0+a} \sum_{m=k}^{\infty} \frac{1}{m} \left(\frac{x_0+a/2-t}{x_0+a/2-t} \right)^m b_l(s) ds \right| |b_j^*(t)| dt \\
&\leq \int_{y_0}^{y_0+b} \left| \int_{x_0}^{x_0+a} \frac{1}{k} \sum_{m=k}^{\infty} \left(\frac{1}{3} \right)^m |b_l(s)| ds \right| |b_j^*(t)| dt \\
&\leq \int_{y_0}^{y_0+b} \int_{x_0}^{x_0+a} \frac{1}{2k \cdot 3^{k-1}} |b_l(s)| ds |b_j^*(t)| dt \\
&\leq \frac{1}{2k \cdot 3^{k-1}} \int_{y_0}^{y_0+b} \sqrt{\int_{x_0}^{x_0+a} b_l(s)^2 ds \int_{x_0}^{x_0+a} 1 ds} |b_j^*(t)| dt \\
&\leq \frac{\sqrt{ab}}{2k \cdot 3^{k-1}}.
\end{aligned}$$

□

Let D be the closed disk of radius $\frac{3}{2}$ centered at $z = \frac{1}{2}$ and L is analytic in a domain containing $D \times D \subset C^2$. Suppose that K is the restriction of L to $[0, 1] \times [0, 1]$. With this kernel K , the following is possible,

Lemma 4.3 *Let B_k and $B_{k'}^*$ be the multiwavelet bases of orders k and k' respectively, where $2^m k = 2^{m'} k'$ and $\frac{k}{k'} = 2^q$ for some nonnegative integer q . Denote the supports I_l of $b_l \in B_k$ and I_j^* of $b_j^* \in B_{k'}^*$ by $[x_0, x_0 + a]$ and $[y_0, y_0 + b]$ respectively and assume that they are separated from the diagonal. Then, for $K_{l,j}^* \equiv \int_{y_0}^{y_0+b} \int_{x_0}^{x_0+a} K(t,s) b_l(s) b_j^*(t) ds dt$, where K is defined above,*

$$|K_{l,j}^*| \leq \frac{\sqrt{ab}}{7 \cdot 8^{k-1}} \sup_{s,t \in \partial D} |K(s,t)|.$$

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