

A Note on the Use of Residual as an Error Estimator for Hammerstein Equations

Hideaki Kaneko Richard D. Noren

Department of Mathematics and Statistics

Old Dominion University

Norfolk, Virginia 23529-0077, USA

A NOTE ON THE USE OF RESIDUAL AS AN ERROR ESTIMATOR FOR HAMMERSTEIN EQUATIONS

H. Kaneko and R. D. Noren

Abstract

In this paper, we show that the residual can be used to estimate the error of a numerical solution for a class of nonlinear Hammerstein equations. It is also shown that the superconvergence of the iterated numerical solution provides a sufficient condition for the residual to be used as an error estimator. Hammerstein equations with smooth as well as weakly singular kernels will be treated.

1 Introduction

One of the most difficult problems that we face when numerically solving a certain operator equation is to discern how close the numerical solution is to the actual solution as the actual solution is not known. In the recent paper [1], Geng, Oden and Demkowicz show that, when the Galerkin method is applied to the Fredholm equations of the second kind, it produces a numerical solution whose residual is asymptotically equivalent to the actual error of the numerical solution. This implies that for sufficiently fine discretizations, the residual can be used as an excellent estimator of the actual error. Golberg and Bowman [2] noticed that the asymptotical equivalence of the residual and the error can be derived by considering the superconvergence of Sloan's iterates. The observation, of course, broadens the scope in which the asymptotical equivalence of the residual and the actual error can be used, since the study of the superconvergence of Sloan's iterates is now well documented [7] and occurs in the classical numerical schemes such as the Galerkin or the collocation schemes when applied to many different equations [2]. In recent papers [4], [5], many results on the superconvergence of Sloan's iterates for the linear Fredholm integral equations were generalized to hold for the nonlinear Hammerstein equations. In addition, the superconvergence of the iterates of the Galerkin as well as the collocation method for the weakly singular Hammerstein equations is discussed in [4] and [5]. We note that the superconvergence results concerning the iterates for the weakly singular Hammerstein equations are new even in the Fredholm equations case. The purpose of this paper is twofold. First, we provide a direct proof that, when the Galerkin method is applied to Hammerstein equations, the corresponding residual and the actual error are asymptotically equivalent, extending the result of [1] to nonlinear equations. Secondly, we utilize the results from [4] and [5] to show that the superconvergence of

the iterates for nonlinear equations provides, as in the linear equation case, a sufficient condition for the asymptotical equivalence of the actual error and the residual.

2 Hammerstein Equations

Consider the following Hammerstein equations,

$$x(t) - \int_0^1 k(t, s)\psi(s, x(s))ds = f(t), \quad 0 \leq t \leq 1, \quad (1.1)$$

where k , f and ψ are known functions and x is the function to be determined. It is well known that the equation of this type arises in the integral reformulation of nonlinear two-point boundary value problems. Define $k_t(s) \equiv k(t, s)$ for $t, s \in [0, 1]$ to be the t section of k . Throughout this paper, we assume the following:

1. $\lim_{t \rightarrow \tau} \|k_t - k_\tau\|_\infty = 0, \quad \tau \in [0, 1];$
2. $f \in C[0, 1];$
3. $\psi(s, x)$ is continuous in $s \in [0, 1]$ and Lipschitz continuous in $x \in (-\infty, \infty)$, i.e., there exists a constant $C_1 > 0$ for which

$$|\psi(s, x_1) - \psi(s, x_2)| \leq C_1|x_1 - x_2|, \quad \text{for all } x_1, x_2 \in (-\infty, \infty);$$

4. the partial derivative $\psi^{(0,1)}$ of ψ with respect to the second variable exists and is Lipschitz continuous, i.e., there exists a constant $C_2 > 0$ such that

$$|\psi^{(0,1)}(t, x_1) - \psi^{(0,1)}(t, x_2)| \leq C_2|x_1 - x_2|, \quad \text{for all } x_1, x_2 \in (-\infty, \infty); \quad (1.2)$$

5. for $x \in C[0, 1]$, $\psi(., x(.))$, $\psi^{(0,1)}(., x(.)) \in C[0, 1]$.

Now we review the Galerkin method for Hammerstein equations. A more detailed account of the method and its superconvergence of the corresponding iterates can be found in [4]. Let n be a positive integer and $\{X_n\}$ be a sequence of finite dimensional subspaces of $C[0, 1]$ such that for any $x \in C[0, 1]$ there exists a sequence $\{x_n\}$, $x_n \in X_n$, for which

$$\|x_n - x\|_\infty \rightarrow 0 \quad \text{as } n \rightarrow \infty. \quad (1.3)$$

Let $P_n: L_2[0, 1] \rightarrow X_n$ be an orthogonal projection for each n . We assume that the projection P_n when restricted to $C[0, 1]$ is uniformly bounded, i.e.

$$P := \sup_n \|P_n|_{C[0,1]}\|_\infty < \infty. \quad (1.4)$$

Then from (1.3) and (1.4), it follows that for each $x \in C[0, 1]$,

$$\|P_n x - x\|_\infty \rightarrow 0, \text{ as } n \rightarrow \infty. \quad (1.5)$$

Now let

$$(K\Psi)(x)(t) \equiv \int_0^1 k(t, s)\psi(s, x(s))ds.$$

With this notation, equation (1.1) takes the following operator form

$$x - K\Psi x = f. \quad (1.6)$$

Similarly the equation that describes the Galerkin method can be given by

$$x_n - P_n K\Psi x_n = P_n f. \quad (1.7)$$

To describe the error of the Galerkin approximation, we introduce operators $\hat{T}$ and T_n by letting

$$\hat{T}x \equiv f + K\Psi x \quad (1.8)$$

and

$$T_n x_n \equiv P_n f + P_n K\Psi x_n \quad (1.9)$$

so that equations (1.6) and (1.7) can be written respectively as $x = \hat{T}x$ and $x_n = T_n x_n$. A proof of the following theorem can be made by directly applying Theorem 2 of Vainikko (1967) [9].

Theorem 2.1 *Let $x_0 \in C[0, 1]$ be an isolated solution of equation (1.6). Assume that 1 is not an eigenvalue of the linear operator $(K\Psi)'(x_0)$, where $(K\Psi)'(x_0)$ denotes the Fréchet derivative of $K\Psi$ at x_0 . Then the Galerkin approximation equation (1.7) has a unique solution $x_n \in B(x_0, \delta)$ for some $\delta > 0$ and for sufficiently large n . Moreover, there exists a constant $0 < q < 1$, independent of n , such that*

$$\frac{\alpha_n}{1+q} \leq \|x_n - x_0\|_\infty \leq \frac{\alpha_n}{1-q}, \quad (1.10)$$

where $\alpha_n \equiv \|(I - T'_n(x_0))^{-1}(T_n(x_0) - \hat{T}(x_0))\|_\infty$. Finally,

$$E_n(x_0) \leq \|x_n - x_0\|_\infty \leq C E_n(x_0), \quad (1.11)$$

where C is a constant independent of n and $E_n(x_0) = \inf_{u \in X_n} \|x_0 - u\|_\infty$.

We are now ready to describe the asymptotic equivalence of the residual and the error of the Galerkin solution. Following Golberg and Bowman [2], we denote the error of the Galerkin approximation by

$$e_n \equiv x_0 - x_n \quad (1.12)$$

and the residual by

$$\delta_n \equiv f - (x_n - K\Psi x_n). \quad (1.13)$$

Lemma 2.2 Denote by x_0 and x_n the solutions of equations (1.6) and (1.7) respectively. Assume that 1 is not an eigenvalue of $(K\Psi)'(x_0)$. Then there exist a sequence μ_n such that $\mu_n \rightarrow 0$ as $n \rightarrow \infty$ and an element η_n between x_0 and x_n for which

$$\|(K\Psi)'(\eta_n)e_n\|_2 \leq \mu_n \|e_n\|_2. \quad (1.14)$$

Proof: By the mean value theorem, there exists η_n between x_0 and x_n such that

$$(K\Psi)'(\eta_n)(e_n) = K\Psi(x_0) - K\Psi(x_n).$$

Using this, we have

$$\begin{aligned} (I - (K\Psi)'(\eta_n)P_n)(K\Psi)'(\eta_n)e_n &= (I - (K\Psi)'(\eta_n)P_n)(K\Psi(x_0) - K\Psi(x_n)) \\ &= (K\Psi(x_0) - K\Psi(x_n)) - (K\Psi)'(\eta_n)(P_n K\Psi(x_0) - P_n K\Psi(x_n)) \\ &= (K\Psi)'(\eta_n)e_n - (K\Psi)'(\eta_n)P_n e_n \\ &= (K\Psi)'(\eta_n)(I - P_n)e_n. \end{aligned}$$

Now since 1 is not an eigenvalue of $(K\Psi)'(x_0)$, $(I - (K\Psi)'(x_0))^{-1}$ exists. Also from (1.2), we have that $\psi^{(0,1)}(t, y)$ is continuous in y uniformly in t . This implies that $(K\Psi)'(y)$ is continuous as a function of y in the space of all bounded linear operators $B(C[0, 1], C[0, 1])$. Since η_n lies between x_n and x_0 , $\eta_n \rightarrow x_0$ as $n \rightarrow \infty$. It follows that $(I - (K\Psi)'(\eta_n))P_n$ converges to $(I - (K\Psi)'(x_0))$ in $B(C[0, 1], C[0, 1])$. Therefore, the existence of the inverse operator $(I - (K\Psi)'(x_0))^{-1}$ guarantees that $((I - (K\Psi)'(\eta_n))P_n)^{-1}$ exists and is uniformly bounded for all sufficiently large n [6]. An $\epsilon/3$ argument also shows that $\lim_{n \rightarrow \infty} \|(K\Psi)'(\eta_n)(I - P_n)\|_2 = 0$. Hence

$$(K\Psi)'(\eta_n)e_n = (I - (K\Psi)'(\eta_n)P_n)^{-1}(K\Psi)'(\eta_n)(I - P_n)e_n,$$

and

$$\|(K\Psi)'(\eta_n)e_n\|_2 \leq \mu_n \|e_n\|_2$$

where $\mu_n \equiv \|(I - (K\Psi)'(\eta_n)P_n)^{-1}\|_2 \|(K\Psi)'(\eta_n)(I - P_n)\|_2 \rightarrow 0$ as $n \rightarrow \infty$. Hence Lemma is proved. $\square$

Before we present our main theorem, the following characterization of the residual is useful.

$$\begin{aligned} \delta_n &= f - (x_n - K\Psi x_n) \\ &= (x_0 - K\Psi x_0) - (x_n - K\Psi x_n) \\ &= (x_0 - x_n) - (K\Psi)'(\eta_n)(x_0 - x_n) \\ &= (I - (K\Psi)'(\eta_n))e_n, \end{aligned} \quad (1.15)$$

Using (1.14) and (1.15), we immediately obtain the following.

Theorem 2.3 *Assume that the conditions in Lemma 2.2 are satisfied. Then the residual δ_n and the error e_n defined respectively in (1.12) and (1.13) are asymptotically equivalent. More specifically, we obtain*

$$\|\delta_n\|_2 = \|e_n - (K\Psi)'(\eta_n)e_n\|_2 \leq (1 + \mu_n)\|e_n\|_2, \quad (1.16)$$

where $\mu_n \rightarrow 0$ as $n \rightarrow \infty$.

We are now in a position to describe Sloan's iterates for the Galerkin approximate solution x_n . For more details, see [4] [7] [8]. Sloan's iterates is defined by

$$\hat{x}_n = f + K\Psi(x_n). \quad (1.17)$$

In the case of linear Fredholm equations, it was proved in [8] that the iterated Galerkin solution $\hat{x}_n$ converges to x_0 faster than the original Galerkin solution x_n to x_0 . The rate of convergence of the iterates can be doubled under certain smoothness conditions imposed upon the kernel and the forcing term. Sloan and Thomee's results were generalized to the nonlinear Hammerstein equations in [4]. The superconvergence of the iterated collocation method for the Fredholm equations is also known [3] and their results were extended to nonlinear equations in [5]. In the papers [4] and [5], in addition to the aforementioned generalizations of superconvergence results to the nonlinear equations, the superconvergence of the iterated Galerkin method and collocation method for weakly singular Hammerstein equations are also discussed. The case for weakly singular equations arises when the kernel $k(t, s)$ in equation (1.1) takes the following form,

$$k(t, s) \equiv g_\alpha(|s - t|) = \begin{cases} |s - t|^{\alpha-1}, & 0 < \alpha < 1, \\ \log |s - t|, & \alpha = 1. \end{cases} \quad (1.18)$$

It is well documented [4] that when the Galerkin or collocation methods are used to approximate the solutions of weakly singular integral equations (linear or nonlinear), the optimal rate of convergence can be achieved if these methods are applied with a class of splines defined over a certain type of nonuniform knots. (See [4] for more details.) The corresponding iterated solutions with kernel described by (1.18) subsequently converges at the rate that is increased by the factor of α . Thus if the splines of order r are used, then we expect that

$$\|\hat{x}_n - x_0\|_\infty = O(h^{r+\alpha}), \quad (1.19)$$

where $h = \max\{t_{i+1} - t_i\}$ with $\{t_i\}$ denoting the nonuniform knots.

Golberg and Bowman [2] proved that the superconvergence of the iterated solution of the Fredholm equation provides a sufficient condition for the conclusion of Theorem 2.3 to hold. The

observation subsequently guarantees the asymptotic equivalence of the residual and the error of the collocation solution in light of the paper [3]. A straightforward analysis shows that the results in [2] can be extended to Hammerstein equations. To this end, we let

$$\hat{e}_n = x_0 - \hat{x}_n.$$

Then

$$\begin{aligned}\delta_n &= f - (x_n - K\Psi x_n) \\ &= \hat{x}_n - x_n \\ &= e_n - \hat{e}_n.\end{aligned}$$

From this, we obtain

$$\left| 1 - \frac{\|\hat{e}_n\|}{\|e_n\|} \right| \leq \frac{\|\delta_n\|}{\|e_n\|} \leq 1 + \frac{\|\hat{e}_n\|}{\|e_n\|}. \quad (1.20)$$

Therefore, the superconvergence of Soaln's iterates, i.e.,

$$\lim_{n \rightarrow \infty} \frac{\|\hat{e}_n\|}{\|e_n\|} = 0 \quad (1.21)$$

gives a sufficient condition for the inequality (1.16) to occur. A plain norm $\|\cdot\|$ above indicates $\|\cdot\|_2$ and $\|\cdot\|_\infty$ for the Galerkin method and for the collocation method respectively. The following theorem summerizes what we have just discussed.

Theorem 2.4 *Let x_0 be an isolated solution of (1.1) and let x_n be its Galerkin (or collocation) approximation to x_0 . Then the actual error e_n and the residual δ_n are asymptotically equivalent, -i.e., $\lim_{n \rightarrow \infty} \frac{\|\delta_n\|}{\|e_n\|} = 1$.*

We have noted that in the case of a smooth kernel k and a forcing term f , if the spline of order r is used in the Galerkin (or collocation) method, then

$$\frac{\|\hat{e}_n\|}{\|e_n\|} = O(h^r),$$

whereas in the case of weakly singular Hammerstein (or Fredholm) equations,

$$\frac{\|\hat{e}_n\|}{\|e_n\|} = O(h^\alpha).$$

This, in turn, provides an order of accuracy of the residual as an error estimator.

References

- [1] P. Geng, J. T. Oden and L. Demkowicz, Numerical solution and a *posteriori* error estimation of exterior acoustics problems by a boundary element method at high wave numbers, *Jl. Acoust. Soc. Am.* Vol. 100(1), (1996), 335-345.
- [2] M. A. Golberg and H. Bowman, Superconvergence and the use of the residual as an error estimator in the BEM-I: theoretical development, *International Jl. of BEM Communications*, Vol. 8, (1997), 230-238.
- [3] I. Graham, S. Joe and I. H. Sloan, Iterated Galerkin versus iterated collocation for integral equations of the second kind, *IMA Jl. Num. Anal.*, 5 (1985), 355-369.
- [4] H. Kaneko and Y. Xu, Superconvergence of the iterated Galerkin methods for Hammerstein equations, *SIAM Jl. Num. Anal.*, Vol. 33 (1996), 1048-1064.
- [5] H. Kaneko, R. D. Noren and P. A. Padilla, Superconvergence of the iterated collocation method for Hammerstein equations, *Jl. Comp. and Appl. Math.*, Vol. 80 (1997), 335-349.
- [6] R. Kress, *Linear Integral Equations*, Springer-Verlag, Berlin (1989).
- [7] I. H. Sloan, Superconvergence, *Numerical Methods for Integral Equations*, Plenum Publ. Co., N.Y., 1990.
- [8] I. H. Sloan and V. Thomee, Superconvergence of the Galerkin iterates for integral equations of the second kind, *Jl. Int. Eqs.*, 9 (1985), 1-23; 22 (1981), 431-438.
- [9] G. Vainikko, Perturbed Galerkin method and general theory of approximate methods for nonlinear equations, *Zh. Vychisl. Mat. Fiz.* 7 (1967), 723-751. Engl. Translation, *U.S.S.R. Comp. Math. and Math. Phys.* 7, No. 4 (1967), 1-41.