Singularity Preserving Galerkin Method For Hammerstein Equations With Logarithmic Kernel

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Abstract

In a recent paper [3], Y. Cao and Y. Xu established the Galerkin method for weakly singular Fredholm integral equations that preserves the singularity of the solution. Their Galerkin method provides a numerical solution that is a linear combination of a certain class of basis functions which includes elements that reflect the singularity of the solution. The purpose of this paper is to extend the result of Cao and Xu and to establish singularity preserving Galerkin method for Hammerstein equations with logarithmic kernel. The iterated singularity preserving Galerkin method is also discussed.

Key words: Singularity preserving Galerkin method, Hammertein equations with logarithmic kernel, Iterated singularity preserving Galerkin method.
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1 Introduction

In this paper, we are concerned with the problem of approximating the solutions of weakly singular Hammerstein equations with logarithmic kernel by the Galerkin method that preserves the singularity of the exact solution. Namely we establish a method that generates an approximate solution in terms of a collection of basis functions some of which are comprised of singular elements that reflect the characteristics of the singularity of the exact solution. The idea of the method originates in the recent paper by Cao and Xu [3]. Cao and Xu studied the special nature of the singularities that are pertinent to the solutions of the weakly singular Fredholm equations of the second kind. Let $C[0,1]$ denote the space of all continuous functions defined on $[0,1]$. The weakly singular Fredholm integral equations of the second kind can be described as

$$y(s) - \int_0^1 g_\alpha(|s-t|)m(s,t)y(t)dt = f(s), \qquad 0 \le s \le 1 \tag{1.1}$$

where $f \in C[0,1]$, m is sufficiently smooth and

$$g_\alpha(|s-t|) = \begin{cases} |s-t|^\alpha, & -1 < \alpha < 0, \\ \log|s-t|, & \alpha = 0, \end{cases} \tag{1.2}$$

where y is, of course, the function to be determined. It is well documented (see, e.g. [19],[16],[4],[23]) that the solutions of the equations described in (1.1) exhibit, in general, mild singularities even in the case of a smooth forcing term f. Here by "mild" singularities, we mean singularities in derivatives. The papers of Richter [16] and Graham [4] contain singularity expansions of the solutions of equation (1.1) in the case of $m(s,t) \equiv 1$. The results of Graham were recently generalized by Cao and Xu for equation (1.1). Information concerning the type of singularities that solutions have is useful when solving equation (1.1) numerically. In order to approximate functions with mild singularities, many investigators utilized the important theorem of Rice [17] that gives an optimal order of approximation to such functions. Rice's theorem is that of non-linear approximation by splines using variable knots. Based upon this idea of approximating the solutions of equation (1.1) by splines defined on nonuniform knots, the collocation method, the Galerkin method and the product-integration method were established for equation (1.1) by Vainikko and Uba [24], by Graham [4] and by Schneider [20] respectively. A modified collocation method was introduced in [13] which also uses the idea of Rice. Recently there has been some considerable interest in the study of the following weakly singular Hammerstein equation:

$$y(s) - \int_0^1 g_\alpha(|s-t|)m(s,t)\psi(t,y(t))dt = f(s), \qquad 0 \le s \le 1 \tag{1.3}$$

where f, m and g_α are defined as in (1.1) and (1.2) and ψ is a known function. It is well known that an equation similar to (1.3) arises naturally as a reformulation to an integral equation of a class of two dimensional Dirichlet problems with certain nonlinear boundary conditions, -e.g., [2]. A study on the regularities of the solution y of equation (1.3) is reported in [10], extending the results of [19]. Subsequently, Kaneko, Noren and Xu used the regularity results to establish the collocation method for weakly singular Hammerstein equations in [11]. The approximate solutions provided by these methods for equations (1.1) and (1.3) are in the form of piecewise polynomials that are not always satisfactory as a tool for approximating functions with singularities. This observation is quite evident in the areas of finite element analysis. Hughes and Akin [7] list several problems (e.g. 'upwind' finite elements for treating convection operators [6],[9],[8]; boundary-layer elements [1] etc.) in which the finite element shape functions are constructed to include polynomials as well as singular functions. Singular shape functions are introduced to the set of basis functions through asymptotic analysis on the solution of the problem that is being considered. It should be pointed out that the analysis involved in the aforementioned papers on the finite element method is centered around the collocation method. The problems such as the choice for the extra collocation points for singular basis elements or the rate of convergence are not addressed in these papers. It should be pointed out that the location of additional collocation points for singular basis elements is critical in detemining the rate of convergence of numerical solutions. A detailed discussion on this subject can be found in [12]. A singularity preserving collocation method, because of the reasons mentioned above, seems to be more difficult to establish.

In this paper, a singularity expansion for the solution of equation (1.3) with logarithmic kernel is given. This extends the results in [10] and [3]. Only the logarithmic kernel is considered here. It is a routine matter, however, to establish, following the argument of Section 2, a singularity expansion for the solution of equation (1.3) with algebraic singularity. The details are left to the reader. The paper is organized as follows: in Section 2, we study the regularity property of the soultion of (1.3) and establish its singularity expansion. The results obtained there generalize the results of [3] and [10]. The singularity expansion is then utilized in Section 3 to achieve the singularity preserving Galerkin method for equation (1.3). Finally, in Section 4, the iterated singularity preserving Galerkin method is discussed, extending some results from [15]. Examples are given in Section 5.

2 Singularity Expansion for Weakly Singular Hammerstein Equations

In this section, we consider the following Hammerstein equation with logarithmic singularity,

$$y(s) - \int_0^1 \log|s-t| m(s,t)\psi(t,y(t))dt = f(s), \qquad 0 \le s \le 1 \tag{2.1}$$

(see (1.3) also). We let

$$K\Psi y(s) \equiv \int_0^1 \log|s-t| m(s,t)\psi(t,y(t))dt. \tag{2.2}$$

Then equation (2.1) can be written in operator form as

$$y - K\Psi y = f. \tag{2.3}$$

Let H^n denote the Sobolev space, $H^n[0,1] = \{w : w^{(n)} \in L_2[0,1]\}$, equipped with the norm $\|w\|_{H^n} = \left(\sum_{i=0}^n \|w^{(i)}\|_2^2\right)^{1/2}$ where $w^{(i)}$ describes the ith generalized derivative of w. We also let $W = W_n$ be the linear space spanned by the functions $s^i \log^j s, (1-s)^i \log^j (1-s); i,j = 1,2,...,n-1$. Throughout this paper, we assume the following conditions:

$$m \in C^{2n}([0,1]\times[0,1]), n \ge 1, \ m \in C^1([0,1]\times[0,1]), n = 0. \tag{2.4}$$

$$\psi \in C^{2n+1}(R \times R) \tag{2.5}$$

$$f \in W \oplus H^n. \tag{2.6}$$

We define

$$Ky(s) \equiv \int_0^1 \log|s-t| m(s,t) y(t) dt. \tag{2.7}$$

First we quote the following result (lemma 4.4(2)) from [3].

Lemma 2.1 *Let $u_1(s) = s^p \log^q s$, and $u_2(s) = (1-s)^p \log^q (1-s)$, for some integers $p, q \geq 1$ and let $f \in H^{n-1}$. Assume that $m \in C^{n+1}([0,1] \times [0,1])$. Then,*

$$(Kf)(s) = \sum_{j=1}^{n-1} \Big[c_j s^j \log s + d_j (1-s)^j \log(1-s) \Big] + v_n(s),$$

$$(Ku_1)(s) = \sum_{j=p+1}^{n-1} \sum_{i=1}^{q+1} c_{ij} s^j (\log s)^i + \sum_{j=q+1}^{n-1} d_j (1-s)^j \log(1-s) + v_n(s),$$

and

$$(Ku_2)(s) = \sum_{j=p+1}^{n-1} \sum_{i=1}^{q+1} c_{ij} (1-s)^j (\log(1-s))^i + \sum_{j=q+1}^{n-1} d_j s^j \log s + v_n(s).$$

Lemma 2.2 *If $u_1(s) = s^p \log^q s, u_2(s) = (1-s)^r \log^l (1-s)$, for some integers $p, q, r, l \geq 1$ are integers, then $u_1 u_2 \in W \oplus H^n$.*

Proof: Expand u_1 in series about $s = 1$ and u_2 about $s = 0$:

$$\begin{aligned} u_1(s) &= \textstyle\sum_{i=0}^{n-1} b_i (1-s)^i + f_1(s), & u_2(s) &= \textstyle\sum_{i=0}^{n-1} a_i s^i + f_2(s), \\ &\equiv P_1(s) + f_1(s) & &\equiv P_2(s) + f_2(s) \end{aligned}$$

where $f_1^{(k)}(s) = O((1-s)^{n-k})$ near $s = 1$, f_1 is analytic at $s = 1$, and $f_1^{(k)} \sim u_1^{(k)}(s) - P_1^{(k)}(0)$ as $s \to 0+$; $f_2^{(k)}(s) = O(s^{n-k})$ near $s = 0$, f_2 is analytic at $s = 0$, and $f_2^{(k)}(s) \sim u_2^{(k)}(s) - P_2^{(k)}(1)$ as $s \to 1^-$.

Now $u_1 u_2 = P_1 P_2 + P_1 f_2 + P_2 f_1 + f_1 f_2$. Clearly $P_1 P_2$ is in H^n. For $f_1 f_2$, we have

$$\frac{d^n}{ds^n}(f_1(s) f_2(s)) = \sum_{i=0}^{n} \binom{n}{i} f_1^{(i)}(s) f_2^{(n-i)}(s).$$

Each term $f_1^{(i)}(s) f_2^{(n-i)}(s), i = 0, 1, ..., n$ satisfies

$$f_1^{(i)}(s) f_2^{(n-i)}(s) = O(f_1^{(i)}(s) t^i) = O([u_1^{(i)}(s) - P_1^{(i)}(0)] s^i) \to 0$$

as $s \to 0+$.

Similarly

$f_1^{(i)}(s)f_2^{(n-i)}(s) \to 0$ as $s \to 1^-$. Thus $f_1 f_2 \in C^n \subseteq H^n$. For f_1P_2 we have $f_1(s)P_2(s) = (u_1(s) - P_1(s))P_2(s) = u_1(s)P_2(s) - P_1(s)P_2(s)$. Since P_2 is a polynomial, $u_1 \in W$, it is easy to see that $u_1P_2 \in W \oplus H^n$ (see $[[3], (4.7)]$). So $f_1P_2 \in H^n$. Similarly $f_2P_1 \in W \oplus H^n$, and Lemma 2 has been verified. □

Lemma 2.3 *A product of an H^n function with a function in W is in $H^n \oplus W$.*

Proof: Let $g \in H^n$ and let u_1 and u_2 be defined as before prior to Lemma 1. For gu_1 we write

$$\begin{aligned} u_1(s)g(s) &= \textstyle\sum_{i=0}^{n-1} \frac{g^{(i)}(0)}{i!} s^{i+p} \log^q s + \frac{s^p \log^q s}{(n-1)!} \int_0^s g^{(n)}(\sigma)(s-\sigma)^{n-1} d\sigma \\ &\equiv T_1 + T_2. \end{aligned}$$

Since $T_1 \in W \oplus H^n$, we turn to T_2 and write

$$\begin{aligned} \frac{d^n T_2}{ds^n} &= \tfrac{1}{(n-1)!} \textstyle\sum_{k=0}^{n} \binom{n}{k} \frac{d^k}{ds^k}[s^p \log^q s] \frac{d^{n-k}}{ds^{n-k}} [\int_0^s g^{(n)}(\sigma)(s-\sigma)^{n-1} d\sigma] \\ &= \tfrac{1}{(n-1)!} \textstyle\sum_{k=1}^{n} \binom{n}{k} \frac{d^k}{ds^k}[s^p \log^q s][(n-1)...k] \int_0^s g^{(n)}(\sigma)(s-\sigma)^{k-1} d\sigma \\ &\quad + s^p \log^q s g^{(n)}(s). \end{aligned}$$

But $s^p \log^q s \in L^\infty, g^{(n)} \in L_2[0,1]$ so $(s^p \log^q s) g^{(n)}(s) \in L^2$.

For the terms

$$b_n(s) \equiv \frac{d^k}{ds^k}[s^p \log^q s] \int_0^s g^{(n)}(\sigma)(s-\sigma)^{k-1} d\sigma$$

we have, for some constant M and nonnegative integer α

$$\begin{aligned} \mid b_n(s) \mid &\le M \tfrac{(-\log s)^\alpha}{s^{k-1}} \textstyle\int_0^s \mid g^{(n)}(\sigma) \mid s^{k-1} d\sigma \\ &= M s(-\log s)^\alpha \tfrac{1}{s} \textstyle\int_0^s \mid g^{(n)}(\sigma) \mid d\sigma. \end{aligned}$$

But $g^{(n)} \in L_2[0,1]$, so by Hardy's inequality [18] (p. 72) $\frac{1}{s}\int_0^s \mid g^{(n)}(\sigma) \mid d\sigma \in L_2[0,1]$. Since $s(-\log s)^\alpha \in L^\infty$ it follows that $b_n \in L_2[0,1]$. Hence $\frac{d^n T_2}{ds^n} \in L_2[0,1]$, or $T_2 \in H^n$, This proves $gu_1 \in W \oplus H^n$.

The case for $gu_2 \in W \oplus H^n$ is similar. □

Finally we need the following:

Lemma 2.4 *The operator $K\Psi$ maps $W \oplus H^n$ into $W \oplus H^{n+1}$.*

Proof: Let $y = w + h,\ w \in W,\ h \in H^n$. We use Taylor's theorem in the form

$$\psi(t,x) = \sum_{k=0}^{n} \frac{1}{k!} \psi^{(0,k)}(t,a)(x-a)^k + \frac{1}{n!} \int_a^x (x-\sigma)^n \psi^{(0,n+1)}(t,\sigma) d\sigma. \tag{2.8}$$

Letting $x = y(s)$ and $a = h(s)$ allows us to write

$$\begin{aligned}(K\Psi)(y)(t) \quad &= \textstyle\sum_{k=0}^{n} \frac{1}{k!} \int_0^1 \log \mid t-s \mid m(t,s)\psi^{(0,k)}(s,h(s))w(s)^k ds \\ &+\textstyle\frac{1}{n!} \int_0^1 \log \mid t-s \mid m(t,s) \int_{h(s)}^{y(s)} \psi^{(0,n+1)}(s,\sigma)(y(s)-\sigma)^n d\sigma ds \\ &\equiv \textstyle\sum_{k=0}^{n} \frac{1}{k!} A_k(t) + \frac{1}{n!} B(t).\end{aligned} \tag{2.9}$$

By (3), $\psi^{(0,k)}(s,h(s)) \in H^n, k = 0,1,...,n$, and by expanding $w(s)^k$ with the multinomial expansion, it is clear that $w(s)^k$ is a sum of terms in W as well as terms of the form $as^p \log^q s(1-s)^r \log^u(1-s)$, $p,q,r,u \geq 1$ are integers. The constant, a, depends on p,q,r and u. Since $\psi^{(0,k)}(h(s)) \in H^n$ and $w(s)^k \in W \oplus H^n, k = 0,1,...,n$, it follows from Lemma 3 that

$$\psi^{(0,k)}(s,h(s))w(s)^k \in W \oplus H^n. \tag{2.10}$$

By Lemma 1 and (2.10), we have

$$A_k \in W \oplus H^{n+1}. \tag{2.11}$$

For $B(t)$, if we prove that

$$F(s) \equiv \int_{h(s)}^{y(s)} \psi^{(0,n+1)}(s,\sigma)(y(s)-\sigma)^n d\sigma \in W \oplus H^n, \tag{2.12}$$

then, also by Lemma 1, $B(t) = K[F](t)$ will be in $W \oplus H^{n+1}$. This will complete the proof of this lemma. First of all, suppose $n \geq 1$. We write

$$F'(s) = -\psi^{(0,n+1)}(s,h(s))w(s)^n h'(s).$$

Since $h \in H^n, \psi \in C^{2n+1}, \psi^{(0,n+1)}(s,h(s)) \in H^n$. By Lemmas 2 and 3, $-\psi^{(0,n+1)}(s,h(s))w(s)^n \in H^n \oplus W$. Since $h' \in H^{n-1}$, it follows that $-\psi^{(0,n+1)}(s,h(s))w(s)^n h'(s) \in H^{n-1} \oplus W$ (Lemma 2). Since $F' \in H^{n-1} \oplus W$ it is clear that $F \in H^n \oplus W$. Second, let $n = 0$. Then $F(s) = \int_{h(s)}^{y(s)} \psi'(s,\sigma)d\sigma = \psi(s,y(s)) - \psi(s,h(s)) \in L_2[0,1] \subseteq W \oplus H^0$.
Thus

$$F \in W \oplus H^n. \tag{2.14}$$

By (2.9), (2.11) and (2.12), it follows that $K\Psi$ maps $W \oplus H^n$ into $W \oplus H^{n+1}$. $\Box$

Using the lemmas which we proved above, we obtain the following main result of this section.

Theorem 2.5 *Suppose the conditions (2.4)-(2.6) hold and y is an isolated solution of (2.1). Then there are constants a_{ij} and b_{ij}, for $i,j = 1,2,...,n-1$, and there is a function v_n in H^n such that*

$$y(t) = \sum_{i=1}^{n-1} \sum_{j=1}^{n-1} [a_{ij}t^i \log^j t + b_{ij}(1-t)^i \log^j(1-t)] + v_n(t). \tag{2.13}$$

Proof: For $n = 0$, this follows from Lemma 4 with $n = 0$. Assume that the result holds for $n = k$, that is, if $f \in H^k \oplus W_k$, then (2.13) holds with $n = k$. Say $y = w_k + v_k$, where $v_k \in H^k, w_k = \sum_{i=1}^{k-1} \sum_{j=1}^{k-1} [a_{ij} t^i \log^j t + b_{ij}(1-t)^i \log^j (1-t)]$.

Now consider the case $n = k+1$ and suppose $f \in H^{k+1} \oplus W_{k+1}$.

Since $y = w_k + v_k$ we write $y = K\Psi y + f = K\Psi(w_k + v_k) + f$. From Lemma 1, $K\Psi(w_k + v_k) \in W_{k+1} \oplus H^{k+1}$. The proof is complete. □

3 Singularity Preserving Galerkin Method

In this section, we establish the singularity preserving Galerkin method for equation (2.1). First we recall the definition of the space of spline functions of order n. Define the partition of $[0,1]$ as

$$\Delta : \ 0 = t_0 < t_1 < ... < t_k < t_{k+1} = 1.$$

Let

$$h = \max_{1 \le i \le k+1} (t_i - t_{i-1}),$$

and assume $h \to 0$ as $k \to \infty$. Denote by Π_n the set of polynomials of degree $n-1$. Then the space of splines of order n with knots t_i's of multiplicity $n-1-\nu$ is defined as

$$S_h^n = S_h^{n,\nu}(\Delta) = \{ s \in C^\nu[0,1] : \ s|_{I_i} \in \Pi_n, \},$$

where $0 \le \nu \le n-1$ and $I_i = (t_{i-1}, t_i)$ for $i = 1, 2, ..., k+1$. It is well known that the dimension of S_h^n is $d = n(k+1) - k(1+\nu)$. S_h^n is spanned by a basis consisting of B-splines $\{B_i\}_{i=1}^d$. We let

$$V_h^n \equiv W \oplus S_h^n \tag{3.1}$$

and denote the orthogonal projection of $L_2[0,1]$ onto V_h^n by P_h^G. The singularity preserving Galerkin method for approximating the solution of equation (2.3) requires the solution $y_h \in V_h^n$ to satisfy the following equation:

$$y_h - P_h^G K\Psi y_h = P_h^G f. \tag{3.2}$$

More specifically, we need to find y_h in the form

$$y_h(s) = \sum_{i,j=1}^{n-1} \alpha_{ij} s^i \log^j s + \sum_{i,j=1}^{n-1} \beta_{ij} (1-s)^i \log^j (1-s) + \sum_{i=1}^{d} \gamma_i B_i(s) \tag{3.3}$$

where $\{\alpha_{ij}, \beta_{ij}\}_{i,j=1}^{n-1}$ and $\{\gamma_i\}_{i=1}^{d}$ are found by solving the following system of nonlinear equations:

$$
\begin{aligned}
\textstyle\sum_{i,j=1}^{n-1} \quad & \alpha_{ij}(s^i \log^j s, s^p \log^q s) + \textstyle\sum_{i,j=1}^{n-1} \beta_{ij}((1-s)^i \log^j (1-s), s^p \log^q s) + \textstyle\sum_{i=1}^{d} \gamma_i(B_i, s^p \log^q s) \\
& -(K\Psi(\textstyle\sum_{i,j=1}^{n-1} \alpha_{ij} s^i \log^j s + \textstyle\sum_{i,j=1}^{n-1} \beta_{ij}(1-s)^i \log^j(1-s) + \textstyle\sum_{i=1}^{d} \gamma_i B_i), s^p \log^q s) \\
& \qquad\qquad = (f, s^p \log^q s) \qquad p, q = 1, 2, \ldots, n-1 \\
\textstyle\sum_{i,j=1}^{n-1} \quad & \alpha_{ij}(s^i \log^j s, (1-s)^p \log^q (1-s)) + \textstyle\sum_{i,j=1}^{n-1} \beta_{ij}((1-s)^i \log^j (1-s), (1-s)^p \log^q (1-s)) \\
& \qquad\qquad + \textstyle\sum_{i=1}^{d} \gamma_i(B_i, (1-s)^p \log^q (1-s)) \\
& -(K\Psi(\textstyle\sum_{i,j=1}^{n-1} \alpha_{ij} s^i \log^j s + \textstyle\sum_{i,j=1}^{n-1} \beta_{ij}(1-s)^i \log^j(1-s) + \textstyle\sum_{i=1}^{d} \gamma_i B_i), \\
& (1-s)^p \log^q (1-s)) = (f, (1-s)^p \log^q (1-s)) \qquad p, q = 1, 2, \ldots, n-1 \\
\textstyle\sum_{i,j=1}^{n-1} \quad & \alpha_{ij}(s^i \log^j s, B_p) + \textstyle\sum_{i,j=1}^{n-1} \beta_{ij}((1-s)^i \log^j (1-s), B_p) + \textstyle\sum_{i=1}^{d} \gamma_i(B_i, B_p) \\
& -(K\Psi(\textstyle\sum_{i,j=1}^{n-1} \alpha_{ij} s^i \log^j s + \textstyle\sum_{i,j=1}^{n-1} b_{i,j}(1-s)^i \log^j(1-s) + \textstyle\sum_{i=1}^{d} \gamma_i B_i), B_p) \\
& \qquad\qquad = (f, B_p) \qquad p = 1, 2, \ldots, d
\end{aligned}
$$

where $(\cdot, \cdot)$ denotes the usual inner product defined on $L_2[0,1]$. Now let P_h be the orthogonal projection of $L_2[0,1]$ onto S_h^n. Then we have

$$P_h v \to v \qquad \text{as } h \to 0 \qquad \text{for all } v \in L_2[0,1]. \tag{3.4}$$

It is well known (e.g. [21]) that if $g \in H^n$, $n \geq 0$, then for each $h > 0$, there exists $\phi_h \in S_h^n$ such that

$$\|g - \phi_h\|_2 \leq C h^n \|g\|_{H^n}, \tag{3.5}$$

where $C > 0$ is a constant independent of h. By virtue of the fact that $P_h u$ is the best L_2 approximation of u from S_h^n, we see immediately that

$$\|P_h u - u\|_2 \leq \|u - \phi_h\|_2 \leq C h^n \|u\|_{H^n}, \text{ for all } u \in H^n. \tag{3.6}$$

The following lemma from [3] is useful in the sequel.

Lemma 3.1 *Let X be a Banach space. Suppose that U_1 and U_2 are two subspaces of X with $U_1 \subseteq U_2$. Assume that $P_1 : X \to U_1$ and $P_2 : X \to U_2$ are linear operators. If P_2 is a projection, then*

$$\|x - P_2 x\|_X \leq (1 + \|P_2\|_X)\|x - P_1 x\|_X \text{ for all } x \in X.$$

For convenience, we introduce operators $\hat{T}$ and T_h by letting

$$\hat{T} y \equiv f + K\Psi y \tag{3.7}$$

and

$$T_h y_h \equiv P_h^G f + P_h^G K\Psi y_h \tag{3.8}$$

so that equations (2.1) and (3.2) can be written respectively as $y = \hat{T}y$ and $y_h = T_h y_h$. The following theorem guarantees the existence of a solution of the singularity preserving Galerkin method (3.2) and describes the accuracy of its approximation.

Theorem 3.2 *Let $y \in L_2[0,1]$ be an isolated solution of equation (2.1). Assume that 1 is not an eigenvalue of the linear operator $(K\Psi)'(y)$, where $(K\Psi)'(y)$ denotes the Fréchet derivative of $K\Psi$ at y. Then the singularity preserving Galerkin approximation equation (3.2) has a unique solution y_h such that $\|y - y_h\|_2 < \delta$ for some $\delta > 0$ and for all $0 < h < h_0$ for some $h_0 > 0$. Moreover, there exists a constant $0 < q < 1$, independent of h, such that*

$$\frac{\alpha_h}{1+q} \leq \|y - y_h\|_2 \leq \frac{\alpha_h}{1-q}, \tag{3.9}$$

where $\alpha_h \equiv \|(I - T_h'(y))^{-1}(T_h(y) - \hat{T}(y))\|_2$. Finally, if $y = w + v$ with $w \in W$ and $v \in H^n$, then

$$\|y - y_h\|_2 \leq Ch^n \|v\|_{H^n}, \qquad \text{whenever } 0 < h < h_0, \tag{3.10}$$

where $C > 0$ is a constant independent of h.

Proof: The existence of a unique solution y_h of equation (3.2) in the disk of radius δ about y and the inequalities in (3.7) can be proved using Theorem 2 of Vainikko [22]. A detailed discussion on this application can be found in [11]. To get (3.10), first we note from Lemma 3.1, for $v \in L_2[0,1]$,

$$\|P_h^G v - v\|_2 \leq (1 + \|P_h^G\|_2)\|P_h v - v\|_2. \tag{3.11}$$

By assumption, $(I - (K\Psi)'(y))^{-1}$ exists. By (3.4), Lemma 3.1 and since $(K\Psi)'(y)$ is a compact linear operator, $\|P_n^G(K\Psi)'(y) - (K\Psi)'(y)\|_2 \to 0$ as $n \to \infty$. Hence $(I - P_n^G(K\Psi)'(y))^{-1} = (I - T_h'(y))^{-1}$ exists and uniformly bounded in $\|\cdot\|_2$ norm. Now, from (3.9),

$$\begin{aligned} \|y - y_h\|_2 &\leq \tfrac{\alpha_h}{1-q} \\ &= \tfrac{1}{1-q}\|(I - T_h'(y))^{-1}(T_h(y) - \hat{T}(y))\|_2 \\ &\leq C\|P_h^G K\Psi y - K\Psi y + P_h^G f - f\|_2 \\ &= C\|P_h^G y - y\|_2. \end{aligned} \tag{3.12}$$

where C is independent of h. Using the uniform boundedness of $\{P_h^G\}$, (3.6), (3.11) and (3.12), we obtain

$$\|y - y_h\|_2 \leq Ch^n \|v\|_{H^n}.$$

□

4 The Iterated Singularity Preserving Galerkin Method

In this section, the superconvergence of the iterated singularity preserving Galerkin method is discussed. We remind the reader that the conditions (2.4), (2.5) and (2.6) are still in effect. The discussion of this section depends heavily upon the recent paper by Kaneko and Xu [15] so that only the points of distinct differences are explained.

Let y_0 be an isolated solution of (2.1). Assume that y_h is the unique solution of (3.2) in the sphere $\|y_0 - y\|_2 \leq \delta$, for some $\delta > 0$. Define

$$y_n^I = f + K\Psi y_h. \tag{4.1}$$

Applying P_h^G to both sides of (4.1), we obtain

$$P_h^G y_h^I = P_h^G f + P_h^G K\Psi y_h. \tag{4.2}$$

Comparing (4.2) with (3.2),

$$P_h^G y_h^I = y_h. \tag{4.3}$$

Substitution of (4.3) into (4.1) yields that y_h^I satisfies the following Hammerstein equation,

$$y_h^I = f + K\Psi P_h^G y_h^I. \tag{4.4}$$

In order to analyse the order of convergence of the iterated singularity preserving Galerkin method for Hammerstein equations with logarithmic singular kernel, we need the following. For any $\epsilon \in R$, let $[0,1]_\epsilon = \{t \in [0,1] : t + \epsilon \in [0,1]\}$. Let Δ_h denote the forward difference operator with step size h. For $\alpha > 0$ and $1 \leq p \leq \infty$, we define the Nikol'skii space $N_p^\alpha[0,1]$ by

$$N_p^\alpha[0,1] = \{x \in L_p[0,1] : |x|_{\alpha,p} := \sup_{h \neq 0} \frac{1}{|h|^{\alpha_0}} \|\Delta_h^2 x^{[\alpha]}\|_{L_p[0,1]_{2h}} < \infty\}, \tag{4.5}$$

where $[\alpha]$ is an integer and $0 < \alpha_0 \leq 1$ are chosen so that $\alpha = [\alpha] + \alpha_0$. $N_p^\alpha[0,1]$ is a Banach space with the norm $\|x\|_{\alpha,p} = \|x\|_p + |x|_{\alpha,p}$. We remark that the function $t^{\alpha-1}$ is in $N_1^\alpha[0,1]$ but is not in $N_1^\beta[0,1]$, for any $\beta > \alpha$, and $\log t \in N_1^1[0,1]$, hence it is in N_1^α for all $0 < \alpha < 1$.

The theorem of Kaneko and Xu [14, Theorem 3.3], with only very minor modification can be written in the following form.

Theorem 4.1 *Let $y_0 \in C[0,1]$ be an isolated solution of equation (2.1) and y_h be the unique solution of (2.5) in the sphere $B(y_0, \delta)$. Let y_h^I be defined by the iterated scheme (4.1). Assume that 1 is not an eigenvalue of $(K\Psi)'(y_0)$. Then, for all $1 \leq p \leq \infty$,*

$$\|y_0 - y_h^I\|_\infty \leq C \left\{ \|y_0 - P_h^G y_0\|_\infty^2 + \sup_{0 \leq t \leq 1} \inf_{u \in V_h^n} \|k(t,.)\psi^{(0,1)}(., y_0(.)) - u\|_q \|y_0 - P_h^G y_0\|_p \right\},$$

where $1/p + 1/q = 1$ and C is a constant independent of n.

As a corollary, we obtain the main result of the section. First, we introduce some notations. Applying the mean-value theorem to $\psi(s,y)$ to get

$$\psi(s,y) = \psi(s,y_0) + \psi^{(0,1)}(s, y_0 + \theta(y - y_0))$$

where $\theta \equiv \theta(s, y_0, y)$ with $0 < \theta < 1$ and $\psi^{(0,1)}$ denotes the partial derivative of ψ with respect to the second variable. Also

$$k(s,t) \equiv \log(|s-t|)m(s,t)$$

and

$$g(s,t,y_0,y,\theta) \equiv k(s,t)\psi^{(0,1)}(s, y_0 + \theta(y - y_0)).$$

Theorem 4.2 *Assume the hypotheses of the previous theorem. Assume also that (2.4)-(2.6) hold. Then*

$$\|y_0 - y_n^I\|_\infty = O(h^{2n}) + O(h^{n+1}) = O(h^{n+1}).$$

Proof: First of all, for each $u \in V_h^n$,

$$\|y_0 - P_h^G y_0\|_\infty \leq \|y_0 - u\|_\infty + \|P_h^G u - P_h^G y_0\|_\infty \leq (1+P)\|y_0 - u\|_\infty, \tag{4.6}$$

where $P \equiv \sup\limits_{h>0} P_h^G < \infty$. Since $y_0 = w + v$ for some $w \in W$ and $v \in H^n$, we let $u = w + u^*$, where $u^* \in S_h^n$. We obtain $\|y_0 - u\|_\infty = \|v - u^*\|_\infty$. With (4.6) this yields

$$\|y_0 - P_h^G y_0\|_\infty \leq (1+P) \inf_{u^* \in S_h^n} \|v - u^*\|_\infty \leq Ch^n. \tag{4.7}$$

The last inequality follows from (3.5). Secondly, by [5], [Theorem 4 (i)], there exists $v_t \in S_h^n$ such that $\|k_t - v_t\|_1 = O(h)$. Since $\nu \geq 1$, $S_h^n = S_h^{n,\nu} \subseteq H^1$, so $v_t \in H^1$.

Since $y_0 \in W \oplus H^n$ it follows that $\psi^{(0,1)}(\cdot, y_0(\cdot)) \in W \oplus H^{n-1}$, by expanding $\psi^{(0,1)}(\cdot, y_0(\cdot))$ in Taylor series centered at v (recall $y_0 = w + v, v \in H^n$) and using (2.10) and (2.12). Consequently, $v_t(\cdot)\psi^{(0,1)}(\cdot, y_0(\cdot)) \in W \oplus H^{n-1}$. Say $v_t(\cdot)\psi^{(0,1)}(\cdot, y_0(\cdot)) = a_t + b_t$, where $a_t \in W$ and $b_t \in H^{n-1}$. Now there exists $u_t \in S_h^n$ such that $\|u_t - b_t\|_1 = O(h^{n-1})$
and

$$\|g_t - u_t - a_t\|_1 \leq \|k_t - v_t\|_1 \|\psi^{(0,1)}(\cdot, y_0(\cdot))\|_\infty + \|v_t(\cdot)\psi^{(0,1)}(\cdot, y_0(\cdot)) - u_t - a_t\|.$$

$$= O(h) + O(h^{n-1}) = O(h),$$

provided $n \geq 2$. Now we apply Theorem 4.1 to get

$$\|y_0 - y_h^I\|_\infty = O(h^{2n}) + O(h^{n+1}) = O(h^{n+1}).$$

5 Numerical Example

Let $m(s,t) = 1$, $g(|s-t|) = \log(|s-t|)$ and $\psi(s,t) = cos(s+t)$ in equation (1.3). We assume f in such a way that $x(t) = \sin t + t\log t$ is the solution. Using splines of order 2 we approximate the solution of the Hammerstein equation with

$$y_0(t) = \sum_{i=1}^{d} \gamma_i B_i$$

and

$$y_1(t) = \sum_{i=1}^{d} \gamma_i B_i + \alpha t \log t + \beta(1-t)\log(1-t) \tag{5.1}$$

y_0 represents the numerical solution that uses only the slpine basis elements whereas y_1 represents the current scheme. y_0 is computed for comparison. Notice that the convergence rate for y_0 is lower due to the logarithmic singularity in the kernel and due to the use of the uniform partition of $[0,1]$. The use of nonuniform partition to obtain the optimal rate of convergence of numerical solution was recently established in [15] for the Galerkin method. It should be pointed out that, as the number of partition points increases, the distribution of these nonuniform points become extremely skewed toward the end points of the interval. This will cause a sensitivity in numerical computations, frequently requiring computations in double precision. An introduction of the singular elements in the basis and working with the uniform partition points will eliminate this problem. The coefficients in (4.18) were obtained by solving thew set of nonlinear equations of Section 3 (immediately following (3.3)) using the Newton-Raphson algorithm. Also, the Gauss-type quadrature algorithm described in [14] is used to calculate all integrals. The computed errors for the spline-only solution and the singularity preserving solution are shown in the following table.

	Errors	
n	y_0	y_1
2	.032756	.004002
3	.018526	.001945
4	.012246	.001147
convergence rate $\approx$	1.4	1.8

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