VARIATIONAL PRINCIPLE AND FIXED POINTS

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Abstract

In this paper, we extend the study that was initiated by Hamel [7] on the relationship among theorems of Ekeland, Caristi and Takahashi. We shall give a new proof of Takahashi's theorem and subsequently a new proof to show that the aforementioned theorems are equivalent which was done in [7]. A series of fixed point theorems for multivalued maps are presented that are equivalent to a well known theorem of Caristi.

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1 Introduction

We begin by listing theorems of Takahashi, Caristi and Ekeland. We give an alternative proof to Takahashi's theorem. The main theorem of [12] is the following ("proper" means not identically equal to ∞):

Theorem 1.1 *(Takahashi [12]) Let (X, d) be a complete metric space and let $\varphi\colon X \to (-\infty, \infty]$ be a proper lower semicontinuous function, bounded from below. Suppose that, for each $u \in X$ with $\varphi(u) > \inf_{x\in X} \varphi(x)$, there is a $v \in X$ such that $v \neq u$ and $\varphi(v) + d(u, v) \leq \varphi(u)$. Then there exists an $x_0 \in X$ such that $\varphi(x_0) = \inf_{x\in X} \varphi(x)$.*

Proof: We let $m = \inf\{\varphi(v)|v \in X\}$, and suppose that $\varphi(u) > m$ for every $u \in X$. We define inductively a sequence $\{u_n\}$ as follows; take $u_0 \in X$ with $\varphi(u_0) < \infty$ and let

$$S_0 = \{v \in X|d(v, u_0) \leq \varphi(u_0) - \varphi(v)\}.$$

$S_0 \neq \emptyset$ since $u_0 \in S_0$, and S_0 is closed since φ is lower semi-continuous. Also, by hypothesis, S_0 is not a singleton set, since $\varphi(u_0) > m$ there is $u_1 \in S_0$, $u_1 \neq u_0$. Clearly $\varphi(u_1) < \varphi(u_0)$ and we are free to choose u_1 such that $\varphi(u_1) = \inf\{\varphi(v)|v \in S_0\}$. This minimum is attained because S_0 is a bounded set and φ is lower semi-continuous, bounded from below.

If $u_0, u_1, \ldots, u_n$ have been chosen, choose $u_{n+1} \in S_n = \{v \in X|d(u_n, v) \leq \varphi(u_n) - \varphi(v)\}$ with $u_{n+1} \neq u_n$, $\varphi(u_{n+1}) < \varphi(u_n)$ and $\varphi(u_{n+1}) = \inf\{\varphi(v)|v \in S_n\}$. We claim that $\{u_n\}$ is Cauchy. To see this, note that $d(u_{n+k}, u_n) \leq \sum_{i=1}^{k} d(u_{n+i}, u_{n+i-1}) \leq \sum_{i=1}^{k}(\varphi(u_{n+i-1}) - \varphi(u_{n+i})) = \varphi(u_n) - \varphi(u_{n+k})$, and since $\varphi(u_n)$ is a decreasing sequence converging to c for some c, for any $\epsilon > 0$, we have $d(u_{n+k}, u_n) < \epsilon$ for all k if n is sufficiently large.

Let $\{u_n\}$ converge to $u \in X$. We make a point here that u_n are all different. We claim that $u \in S_n$ for every n. We have

$$\begin{aligned} d(u_{n-k}, u_n) &\leq \textstyle\sum_{i=0}^{k-1} d(u_{n-k+i}, u_{n-k+i+1}) \\ &\leq \textstyle\sum_{i=0}^{k-1} [\varphi(u_{n-k+1}) - \varphi(u_{n-k+i+1})] \\ &= \varphi(u_{n-k}) - \varphi(u_n), \end{aligned}$$

which shows that $u_n \in S_{n-k}$, for $k = 1, 2, \ldots, n$, so that $u_n \in \cap_{i=0}^{n-1} S_i$. We thus have, for $k > n$, $u_k \in \cap_{i=0}^{n} S_i$, and since $\cap_{i=0}^{n} S_i$ is closed, $u \in \cap_{i=0}^{n} S_i$. Thus, in particular, $u \in S_n$, and since $u \neq u_n$, we get $\varphi(u) < \varphi(u_n)$. This contradicts the choice of u_n in S_n. This completes the proof. $\square$.

We note that the formulation that Hamel uses to describe Takahashi's condition is the following:

Condition (T): There exists an $\alpha > 0$ such that for each $u \in X$ such that $\varphi(u) > \inf_{x \in X} \varphi(x)$, there is a $v \in X$ such that $v \neq u$ and $\varphi(v) + \alpha d(u, v) \leq \varphi(u)$.

Condition (T) is different from the condition that was originally given by Takahashi. The condition used in Theorem 1.1 is the original definition of Takahashi's condition. It should be pointed out that condition (T) encompasses a wider class of lower semicontinuous functions than the condition in Theorem 1.1. For example, consider $f\colon R \to R$ defined by $f(x) = 1$ for $x \in (0, 4)$ and $f(x) = 0$ elsewhere. The function f does not satisfy the condition in Theorem 1.1: for $x = 2$, there is no y such that $|y - 2| \leq 1 - f(y)$ holds. However, condition (T) is satisfied by f by the metric $\alpha d(x, y) = \frac{1}{2}|x - y|$.

Takahashi [12] observed that Theorem 1.1 includes as corollaries three well known theorems. They are the fixed point theorem of Caristi [2], the ϵ-variational principle of Ekeland [6] and the fixed point theorem of Nadler [9]. Caristi's theorem is the following:

Theorem 1.2 *(Caristi [2]) Let X be a complete metric space and let $\varphi\colon X \to R$ be a lower semicontinuous function bounded from below. Let $T\colon X \to X$ be a mapping satisfying*

$$d(x, Tx) \le \varphi(x) - \varphi(Tx) \tag{1.2}$$

for every $x \in X$. Then there exists an $x_0 \in X$ with $Tx_0 = x_0$.

Much attention was drawn to Caristi's theorem since its publication because it requires no continuity on the mapping T. It also contains as a special case the fixed point theorem for multivalued contractions of Nadler [9] that is a generalization of the classical Banach fixed point principle. Nadler proved that if T is a mapping of a complete metric space X to $CB(X)$(=the family of all closed bounded subsets of X) that satisfies

$$H(Tx, Ty) \le kd(x, y) \qquad \text{for all } x, y \in X,\ 0 \le k < 1,$$

then T has a fixed point in X. Recently, some work has been made toward generalizing this theorem by relaxing the condition on the contractive constant k. In [11] (p.40), Reich proved that a mapping $T\colon X \to K(X)$(=the family of all compact subsets of X) has a fixed point in X if it satisfies $H(Tx, Ty) \le k(d(x, y))d(x, y)$ for all $x, y \in X$, where $k\colon (0, \infty) \to [0, 1)$ satisfies $\limsup_{r \to t^+} k(r) < 1$ for every $t \in (0, \infty)$. This result generalizes the fixed point theorem for single-valued mappings that was proved by Boyd and Wong [1]. Some attempts [3, 5, 8] were made to replace $K(X)$ by $CB(X)$ thereby generalizing Reich's theorem. Theorem 1.3 of Caristi, although it is of extremely general character, does not seem to have an immediate relationship with these theorems.

Now we return to the main scope of this paper and state the variational principle of Ekeland.

Theorem 1.3 *(Ekeland [6]) Let (X, d) be a complete metric space and $\varphi\colon X \to (-\infty, \infty]$ a proper lower semicontinuous function bounded from below. Let $\epsilon > 0$ be given and a point $u \in X$ such that*

$$\varphi(u) \le \inf_{x \in X} \varphi(x) + \epsilon.$$

Then there exists some point $v \in X$ such that

$$\begin{aligned} \varphi(v) &\le \varphi(u) \\ d(u, v) &\le 1 \\ \varphi(w) &> \varphi(v) - \epsilon d(v, w) \qquad \text{for all } w \ne v. \end{aligned}$$

A number of useful applications of Theorem 1.3 are described in [6]. It is well documented that Theorems 1.2 and 1.3 are equivalent, e.g. see [12],[7]. In [7], Hamel observed that Theorem 1.1 can be derived from Theorem 1.2, making Theorems 1.1, 1.2 and 1.3 equivalent. The main purpose of Section 2 is to provide the reader with additional equivalent formulations of these theorems. This provides us with an alternative perspective to the approach of Hamel in [7]. In addition, a series of fixed point theorems are given for lower semicontinuous multifunctions. In the final section, Section 3, the argument used to prove Theorem 1.1 is used to prove the existence of the weak sharp minima for a class of lower semicontinuous functions. This confirms and expands the result obtained in theorem 2 (ii) of [7].

2 Variational Principle and Fixed Point Theorems

First, in this section, we make an observation that the sequence $\{u_n\}$ generated in the proof of theorem 1.1 [12] actually converges to a minimizer of φ. Theorem 2.1 below is Theorem 1.1 slightly modified to reflect the observed point above. The proof is essentially the one given in [12].

Theorem 2.1 *Let (X,d) be a complete metric space and let $\varphi\colon X \to (-\infty,\infty]$ be a proper lower semicontinuous function, bounded from below. Suppose that, for each $u \in X$ with $\varphi(u) > \inf_{x\in X}\varphi(x)$, there is a $v \in X$ such that $v \neq u$ and $\varphi(v) + d(u,v) \leq \varphi(u)$. Then a sequence $\{u_n\}$ can be constructed that converges to a minimizer x_0 of φ, i.e., $x_0 \in X$ is such that $\varphi(x_0) = \inf_{x\in X}\varphi(x)$.*

Proof: Let $u_0 \in X$. If $\varphi(u_0) = \inf_{x\in X}\varphi(x) \equiv c$, then we are done. If $\varphi(u_i) > c$ for $i = 0, 1, \ldots, n-1$ $(n \geq 1)$, then find $u_n \in S_n$ where

$$S_n = \{w \in X\colon \varphi(w) + d(u_{n-1}, w) \leq \varphi(u_{n-1})\}$$

such that

$$\varphi(u_n) \leq \inf_{w\in S_n}\varphi(w) + \frac{1}{2}\{\varphi(u_{n-1}) - \inf_{w\in S_n}\varphi(w)\}.$$

Arguing as in [12], we see that $\{u_n\}$ is a Cauchy sequence and that, with $u_n \to x_0 \in X$,

$$d(u_n, x_0) \leq \varphi(u_n) - \varphi(x_0).$$

We claim that x_0 is a minimizer for φ. If not, by hypothesis, there exists $z \in X$, $z \neq x_0$ such that

$$\begin{aligned}\varphi(z) &\leq \varphi(x_0) - d(x_0, z)\\ &\leq \varphi(x_0) - d(x_0, z) + \varphi(u_n) - \varphi(x_0) - d(u_n, x_0)\\ &\leq \varphi(u_n) - d(u_n, z).\end{aligned}$$

Hence $z \in S_n$. The definition of S_n implies that

$$2\varphi(u_n) - \varphi(u_{n-1}) \leq \inf_{w \in S_n} \varphi(w) \leq \varphi(z).$$

Using the above inequalities, we obtain the following contradiction,

$$\varphi(z) < \varphi(x_0) \leq \varphi(z).$$

Hence x_0 is a minimizer of φ. $\square$

Takahashi demonstrated the generality of Theorem 1.1 by including as corollaries three well known theorems. They are the fixed point theorem of Caristi [2], the ϵ-variational principle of Ekeland [6] and the fixed point theorem of Nadler [9]. Our next task is to show that Theorem 1.1 and Theorem 1.3 of Caristi are indeed equivalent, providing an alternative proof to Theorem 1 of [7].

Proposition 2.2 *Theorem 1.1 of Takahashi and Theorem 1.2 of Caristi are equivalent. Hence they are equivalent to Theorem 1.3 of Ekeland.*

Proof: As was stated before this Proposition, Takahashi [12] showed that his Theorem 1.1 contains Theorem 1.2 as a corollary. For a converse, suppose that there is no $x_0 \in X$ such that $\varphi(x_0) = \inf_{x \in X} \varphi(x)$. Now define $S\colon X \to 2^X \setminus \emptyset$ by $Sx = \{y\colon \varphi(y) + d(x, y) \leq \varphi(x)\}$ and $T\colon X \to X$ by $Tx \in Sx \setminus \{x\}$. This is possible by the hypothesis to Takahashi's theorem. But T is fixed point free, and this is impossible by Caristi's theorem. Hence there is $x_0 \in X$ with $\varphi(x_0) = \inf_{x \in X} \varphi(x)$.

It is well known that Theorems 1.2 and 1.3 are equivalent [12]. $\square$

We conclude this section by describing four other formulations in terms of the fixed points of lower semicontinuous multifunctions that can be shown to be equivalent to Theorem 1.1 of Takahashi. A similar development was made by Park [10].

Theorem 2.3 *Let X be a complete metric space and let $\varphi\colon X \to (-\infty, \infty]$ be a proper lower semicontinuous function bounded from below. Let $T\colon X \to 2^X \setminus \{\emptyset\}$. Suppose that, for all $x \notin T(x)$, there exists $y \neq x$ that satisfies*

$$\varphi(y) + d(y, x) \leq \varphi(x).$$

Then T has a fixed point in X.

Theorem 2.4 *Let X be a complete metric space and let $\varphi\colon X \to (-\infty, \infty]$ be a proper lower semicontinuous function bounded from below. Let $T\colon X \to 2^X \setminus \{\emptyset\}$ be such that for all $x \in X$ with $T(x) \neq \emptyset$ and for all $y \in T(x)$,*

$$\varphi(y) + d(y, x) \leq \varphi(x).$$

Then there exists $u \in X$ such that $T(u) = \{u\}$.

Theorem 2.5 *Let X be a complete metric space and let $\varphi\colon X \to (-\infty, \infty]$ be a proper lower semicontinuous function bounded from below. Let $T : X \to 2^X \setminus \{\emptyset\}$ be such that for all $x \in X$, there exists $y \in T(x)$ such that*

$$d(y, x) \leq \varphi(x) - \varphi(y).$$

Then T has a fixed point in X.

Theorem 2.6 *Let X be a complete metric space and let $\varphi\colon X \to (-\infty, \infty]$ be a proper lower semicontinuous function bounded from below. Let $T\colon X \to 2^X \setminus \{\emptyset\}$ be closed. Suppose that*

$$d(x, T(x)) \leq \varphi(x) - \sup_{y \in T(x)} \varphi(y) \quad \text{for } x \in X.$$

Then T has a fixed point in X.

The main result of the section is the following:

Proposition 2.7 *Theorems 2.3, 2.4, 2.5 2.6 and 1.1 are equivalent.*

Proof: Theorem 1.1 $\Rightarrow$ Theorem 2.3

Suppose that T has no fixed point. Then for all $x \in X$, $x \notin T(x)$ so that by assumption there exists $y \neq x$ such that $\varphi(y) + d(y, x) \leq \varphi(x)$. Then by Theorem 1.1, there exists $x_0 \in X$ such that $\varphi(x_0) = \inf_{x \in X} \varphi(x)$. Let $y_0 \in T(x_0)$ be such that $y_0 \neq x_0$ and $\varphi(y_0) + d(y_0, x_0) \leq \varphi(x_0)$. Then

$$0 < d(x_0, y_0) \leq \varphi(x_0) - \varphi(y_0) \leq \varphi(y_0) - \varphi(y_0) = 0.$$

This contradiction proves the implication.

Theorem 2.3 $\Rightarrow$ Theorem 2.4

Suppose that there is no element $u \in X$ for which $T(u) = \{u\}$. For each $x \in X$, define f by $f(x) \in T(x) \setminus \{x\}$ so that f is fixed point free. Then $\varphi(f(x)) + d(f(x), x) \leq \varphi(x)$. By Theorem

3.3, T must have a fixed point and it must be a fixed point of f also by Theorem 1.3 of Caristi. This contradiction proves the impication.

Theorem 2.4 $\Rightarrow$ Theorem 1.1

Define $T: X \to 2^X$ by $T(x) = \{y: \varphi(y) + d(x,y) \leq \varphi(x)\}$. Suppose that there is no $x_0 \in X$ such that $\varphi(x_0) = \inf_{x \in X} \varphi(x)$. By the assumption in Theorem 1.1, $T(x) \neq \emptyset$. Then for each $x \in X$ and $y \in T(x)$, $\varphi(y) + d(y,x) \leq \varphi(x)$. Hence by Theorem 3.4, there exists $u \in X$ such that $T(u) = \{u\}$. This shows that there is no $v \neq u$ for which $\varphi(v) + d(u,v) \leq \varphi(u)$.

Theorem 2.4 $\Rightarrow$ Theorem 2.5

This is obvious, since the condition in Theorem 2.4 implies the condition in Theorem 2.5.

Theorem 2.5 $\Rightarrow$ Theorem 2.6

Let $\psi(x) \equiv \frac{1}{2}\varphi(x)$. Let $x \in X$. If $d(x, T(x)) = 0$, then $x \in T(x)$ since $T(x)$ is closed in X. Thus if T has no fixed point, then $d(x, T(x)) > 0$ for any $x \in X$. Let $y \in T(x)$ be such that $d(x,y) < \frac{1}{2}d(x, T(x))$. Then

$$d(x,y) \leq \frac{1}{2}d(x, T(x)) \leq \frac{1}{2}(\varphi(x) - \sup_{y \in T(x)} \varphi(y)) \leq \psi(x) - \psi(y).$$

Since ψ is a proper lower semicontinuous function bounded from below, by Theorem 2.5, T has a fixed point.

Theorem 2.6 $\Rightarrow$ Theorem 2.4

Note that $T(x) \equiv \{y\}$ is closed. Hence Theorem 2.4 is a special case of Theorem 2.6.

This completes the proof of Proposition 2.7.

□

3 Application to Weak Sharp Minima

In this section, we make use of the argument employed to demonstrate Theorem 1.1 to prove the existence of weak sharp minima for a class of lower semicontinuous functions. This expands theorem 2 (ii) of Hamel by giving an alternative approach. As before, let X be a complete metric space and $\varphi: X \to (-\infty, \infty]$ be lower semicontinuous. We define

$$m \equiv \inf\{\varphi(u) | u \in X\}$$

and

$$M \equiv \{v \in X | \varphi(v) = m\}. \tag{3.1}$$

Then we say that φ has weak sharp minima if, for any $u \in X$, we have

$$d(u, M) \leq \varphi(u) - m.$$

Hamel begins section 2 of his paper [7] by stating that "we want to characterize functions which satisfy the condition of Takahashi" and that

Theorem 3.1 *(Theorem 2 [7]) Let X be a complete metric space and $\varphi\colon X \to (-\infty, \infty]$ be a lower semicontinuous function that is bounded from below.*

(i) If there exists an $\alpha > 0$ and a minimizer $u \in X$ of the function φ such that

$$\varphi(v) - \varphi(u) \geq \alpha d(v, u), \qquad \text{for all } v \in X, \tag{1}$$

then f satisfies the condition of Takahashi (as in condition (T) defined earlier) with same $\alpha > 0$.

(ii) Suppose that φ satisfies condition (T) with $\alpha > 0$. Then

$$\varphi(u) - \varphi(v) \geq \alpha dist(x, M), \qquad \text{for all } u \in X \text{ and } v \in M. \tag{2}$$

We note that condition (T) does indeed imply (2) above, but (2) does not in turn imply (1). The fact that (2) can hold without (1) holding may be seen by letting $f\colon R \to R$ be defined by $f(x) = 0$ for all $x \in R$. Hence theorem 2 of [7] does not provide a complete characterization of functions which satisfy condition (T). Our final goal of this paper is to provide a new proof of the existence of weak sharp minima for lower semicontinuous functions that satisfy the condition of Takahashi (the condition described in Theorem 1.1).

Theorem 3.2 *(compare to Theorem 3.1 (ii)) Let X be a complete metric space and $\varphi\colon X \to (\infty, \infty]$ be a lower semicontinuous function that is bounded from below. Suppose that, for any $u \in X$ with $\inf_{x \in x} \varphi(x) < \varphi(u)$, there exists $v \in X$ such that $v \neq u$ and*

$$d(u, v) \leq \varphi(u) - \varphi(v).$$

Then M defined in (3.1) is nonempty and φ has weak sharp minima.

Proof: For $u \in X$, define

$$A(u) \equiv \{v \in X | d(u, v) \leq \varphi(u) - \varphi(v)\}.$$

Since φ is lower semicontinuous, $A(u)$ is a closed set. By Theorem 1.1, $M \neq \emptyset$, where M is defined in (3.1). Note that $\varphi(v) \leq \varphi(u)$ for every $v \in A(u)$. Now, by way of contradiction, let us assume that there is $u_0 \in X$ with

$$\varphi(u_0) - m < d(u_0, M). \tag{3.2}$$

Clearly, $u_0 \notin M$ and this is true for every $v \in A(u_0)$. For if there were $v \in A(u_0)$ with $\varphi(v) = m$, then we get $d(u_0, M) \le d(u_0, v) \le \varphi(u_0) - m$, which contradicts (3.2). We also note that (3.2) holds for every $v \in A(u_0)$. To see this, take $v \in A(u_0)$, $w \in M$, so that $d(u_0, w) \le d(u_0, v) + d(v, w) \le \varphi(u_0) - \varphi(v) + d(v, w)$ and this yields $d(u_0, M) \le \varphi(u_0) - \varphi(v) + d(v, M)$. But $d(u_0, M) > \varphi(u_0) - m$, from (3.2), and together this gives $\varphi(u_0) - m < \varphi(u_0) - \varphi(v) + d(v, M)$, which is $\varphi(v) - m < d(v, M)$, giving (3.2) with v in place of u_0.

Since $\varphi(u_0) > m$, by hypothesis there is $u_1 \in A(u_0)$ with $u_1 \neq u_0$. Since $\varphi(u_1) - m < d(u_1, M)$, again it is clear that $u_1 \notin M$, that $\varphi(u_1) < \varphi(u_0)$ and that we can again show as above that $\varphi(v) - m < d(v, M)$ for every $v \in A(u_1)$ and $A(u_1) \cap M = \emptyset$. In addition, we select u_1 such that $\varphi(u_1) = \inf\{\varphi(v) | v \in A(u_1)\}$. This is possible since X is complete, φ is lower semicontinuous, and $A(u_1)$ is closed and nonempty. Continuing in this way, we generate a sequence $\{u_n\}$ with the above properties. Namely, if $u_0, u_1, \ldots, u_n$ have been chosen so that at $u_i \in A(u_{i-1})$, $\varphi(u_i) < \varphi(u_{i-1})$, $\varphi(u_i) = \inf\{\varphi(v) | v \in A(u_i)\}$, $A(u_{i-1}) \cap M = \emptyset$, $i = 1, 2, \ldots, n$ and $\varphi(v) - m < d(v, M)$ for every $v \in \cup_{i=1}^{n} A(u_{i-1})$, then, since $u_n \notin M$, we can choose $u_{n+1} \in A(u_n)$, $u_{n+1} \neq u_n$, with $\varphi(u_{n+1}) = \inf\{\varphi(v) | v \in A(u_n)\}$, and as above we will have $\varphi(u_{n+1}) > m$, $\varphi(u_{n+1}) < \varphi(u_n)$ and $\varphi(v) - m < d(v, M)$ for each $v \in A(u_{n+1})$. To see the latter, we write again, just as above, $\varphi(u_n) - m < d(u_n, M) \le \varphi(u_n) - \varphi(u_{n+1}) + d(u_{n+1}, M)$, giving $\varphi(u_{n+1}) - m < d(u_{n+1}, M)$. Hence, $A(u_{n+1}) \cap M = \emptyset$.

We now have our sequence $\{u_n\}$ consisting of all different elements and $\varphi(u_{n+1}) < \varphi(u_n)$. Since $d(u_{n+k}, u_n) \le \sum_{i=1}^{k} d(u_{n+i}, u_{n+i-1}) \le \sum_{i=1}^{k} (\varphi(u_{n+i-1}) - \varphi(u_{n+i})) = \varphi(u_n) - \varphi(u_{n+k})$, and noting that $\varphi(u_n)$ monotonically decreases to some c, $\{u_n\}$ must be Cauchy. Let u_n converge to $u \in X$. We now show that $u \in \cap_{i=0}^{\infty} A(u_i)$. We first show that, for every n, $u_n \in \cap_{i=0}^{n-1} A(u_i)$. This follows from the following;

$$
\begin{aligned}
d(u_{n-k}, u_n) &\le \textstyle\sum_{j=0}^{k-1} d(u_{n-k+j}, u_{n-k+j+1}) \\
&\le \textstyle\sum_{j=0}^{k-1} [\varphi(u_{n-k+j}) - \varphi(u_{n-k+j+1})] \\
&= \varphi(u_{n-k}) - \varphi(u_n),
\end{aligned}
$$

which shows (recall that all the u_i are outside M) that $u_n \in A(u_{n-k})$, $k = 1, \cdots, n$, hence $u_n \in \cap_{i=0}^{n-1} A(u_i)$. It follows immediately from this that $u_k \in \cap_{i=0}^{n-1} A(u_i)$ for all $k \ge n$. Since $\cap_{i=0}^{n-1} A(u_i)$ is a closed set, $u \in \cap_{i=0}^{\infty} A(u_i)$. Thus $u \in A(u_n)$, and $u \neq u_n$; hence $\varphi(u) < \varphi(u_n)$, and this is a contradiction, since $\varphi(v) \ge \varphi(u_n)$ for every $v \in A(u_n)$. $\square$.

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