A Construction of Wavelets

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Abstract

In this note, given a multiresolution analysis, we construct a class of scaling functions with four coefficient scaling filters using only elementary algebraic operations.

1 A Review of Construction of Wavelets

We begin this note by recalling the definition of a multiresolution analysis.

Definition A *multiresolution analysis* $\cdots \subseteq V_{-1} \subseteq V_0 \subseteq V_1 \subseteq \cdots$ with scaling function φ is an increasing sequence of subspaces of $L^2(R)$ satisfying the following four conditions:

1. (density) $\cup_j V_j$ is dense in $L^2(R)$,
2. (separation) $\cap_j V_j = \{0\}$,
3. (scaling) $f(x) \in V_j \Leftrightarrow f(2^{-j}x) \in V_0$,
4. (orthonormality) $\{\varphi(x-\gamma)\}_{\gamma \in Z}$ is an orthonormal basis for V_0.

First of all, $\{2^{j/2}\varphi(2^j x - \gamma)\}_{\gamma \in Z}$ forms an orthonormal basis for V_j. This is evident from the definition. In order to form an orthonormal basis for $L^2(R)$, the density condition 1 seems to suggest, at first, to combine all the orthonormal bases $\{2^{j/2}\varphi(2^j x - \gamma)\}_{\gamma \in Z}$ of V_j. But this does not work since there are distinct elements from two orthonormal bases, $\{2^{j/2}\varphi(2^j x - \gamma)\}_{\gamma \in Z}$ for V_j and $\{2^{(j+1)/2}\varphi(2^{j+1} x - \gamma)\}_{\gamma \in Z}$ for V_{j+1}, that are not orthogonal to each other. What is required at this point is the construction of an orthonormal basis for the orthogonal complement W_0 of V_0 in V_1. More generally, we need to find an orthonormal basis for W_j where

$$V_{j+1} = V_j \oplus W_j,$$

for $j = 0, 1, \ldots$ The function ψ for which $\{\psi(x-\gamma)\}_{\gamma \in Z}$ is an orthonormal basis for W_0 is the *wavelet generator* . Once ψ is found, then $\{2^{j/2}\psi(2^j x - \gamma)\}_{\gamma \in Z}$ form an orthonormal basis for W_j and as

$$L^2(R) = V_0 \oplus (\oplus_{j=0}^{\infty} W_j) = \oplus_{j \in Z} W_j,$$

$\{\varphi(x-\gamma)\}_{\gamma\in Z}\cup\{2^{j/2}\psi(2^jx-\gamma)\}_{\gamma\in Z,j\geq 0}$ or $\{2^{j/2}\psi(2^jx-\gamma)\}_{\gamma\in Z,j\geq 0}$ form an orthonormal wavelet basis for $L^2(R)$.

Since $\varphi\in V_0\subseteq V_1$, we must have the following *scaling identity* ,

$$\varphi(x)=\sum_{\gamma\in Z}a_\gamma\varphi(2x-\gamma). \tag{1.1}$$

It is well known [1] that the conditions that must be met by the coefficients a_i's are the following:

$$\sum_{\gamma\in Z}|a_\gamma|^2=2, \tag{1.2}$$

$$\sum_{\gamma'\in Z}a_{\gamma'}a_{2\gamma+\gamma'}=2\delta(\gamma,0), \tag{1.3}$$

and

$$\sum_{\gamma\in Z}a_\gamma=2. \tag{1.4}$$

Equation (1.2) is a consequence of the scaling identity and of the fact that $\|\varphi\|_2=1$. Equation (1.3) is obtained from condition 4 of the definition, -i.e.,

$$\int\varphi(x-\gamma)\varphi(x)\,dx=\delta(\gamma,0),$$

upon substituting the scaling identities for $\varphi(x-\gamma)$ and $\varphi(x)$. Equation (1.4) is obtained also from the scaling identity with additional condition that $\int\varphi(x)dx\neq 0$. Namely, integrate both sides of (1.1) and make a change of variable. Note that (1.2) is a special case of (1.3). Clearly, the scaling function determines a multiresolution analysis, but not conversely. A construction of wavelets involves reversing the procedure. In other words, we need a characterization of a function that satisfies the scaling identity (1.1) and that generates a multiresolution analysis. In order to better explain the content of the present note which is described in the next section, let us review below in five steps how a general construction of wavelets is done. We refer the reader to [1] and [3] for a more complete descriptions of these steps.

Step 1: This step consists of producing solutions to algebraic identities (1.2)-(1.4).

Step 2: Once a_i's are determined, a possible scaling function φ is defined. This can be done, for instance, by finding a fixed point of the linear transformation

$$Sf(x)=\sum_{\gamma\in Z}a_\gamma f(2x-\gamma)$$

by iterations

$$\varphi = \lim_{n\to\infty} S^n f$$

with an appropriate initial function f.

Step 3: Now, we must verify that the function φ, defined in Step 2, generates a multiresolution analysis. To this end, we let

$$A_0(\xi) = \frac{1}{2}\sum_{\gamma\in Z} a_\gamma e^{2\pi i\gamma\xi}.$$

It turns out that the following condition (1.5) along with (1.2)-(1.4) serve as sufficient conditions for the orthonomality of $\{\varphi(x-\gamma)\}_{\gamma\in Z}$.

$$A_0(\xi) \neq 0 \qquad for \;\; |\xi| \leq \frac{1}{4}. \tag{1.5}$$

Step 4: This step is to study those conditions that are necessary to construct wavelets. Let $\psi_0 = \varphi$ and $\psi_1 = \psi$, a wavelet generator to be determined. Also, let $a^0_\gamma = a_\gamma$, the solutions of (1.2)-(1.4). Since $\{\psi_k(x-\gamma)\}_{\gamma\in Z, k=0,1}$ must be an orthonormal basis for $V_1 = V_0 \oplus W_0$, noting that $\{\varphi(2x-\gamma)\}_{\gamma\in Z}$ is an orthonormal basis for V_1,

$$\psi_k(x) = \sum_{\gamma\in Z} a^k_\gamma \varphi(2x-\gamma), \qquad k = 0, 1. \tag{1.6}$$

Equation (1.6) with $k = 0$ is the scaling identity (1.1). The condition that $\psi_0 \perp \psi_1$, or more precisely, with $j, k \in \{0, 1\}$, $\gamma \in Z$,

$$\int \psi_j(x-\gamma)\psi_k(x)dx = \delta(j,k)\delta(\gamma,0),$$

leads to, upon replacing ψ_0 and ψ_1 by the corresponding expressions in (1.6),

$$\sum_{\gamma'\in Z} a^j_{\gamma'} a^k_{2\gamma+\gamma'} = 2\delta(j,k)\delta(\gamma,0). \tag{1.7}$$

Equation (1.7) contains (1.2) and (1.3) as special cases. We can not expect the wavelet generator ψ to satisfy $\int \psi(x)dx \neq 0$, so we do not have an equivalent condition to (1.4) for a^1_γ. Hence, we can only use (1.4), restated below in the current notation as

$$\sum_{\gamma\in Z} a^0_\gamma = 2. \tag{1.8}$$

At this point, we have reduced the problem of constructing wavelets to the solution of algebraic identities (1.7) and (1.8), together with condition (1.5), which was to guarantee the orthonomality condition 4 for φ. In summary, we have (Theorem 4.3, [3]) that

Theorem 1.1 *Suppose φ generates a multiresolution analysis and a_γ^k satisfy (1.7) and (1.8) with ψ_k defined by (1.6) and $\psi_0 = \varphi$. Then the functions $\{2^{j/2}\psi_1(2^j x - \gamma)\}$ for $j \in Z$, $\gamma \in Z$ form an orthonormal basis of $L^2(R)$.*

Step 5: This step completes the task that began in Step 4. Namely, we need to produce the solutions of (1.7) and (1.8) that satisfy (1.5). The function A_0 defined earlier can be written as

$$A_0(\xi) = \frac{1}{2}\sum_{\gamma \in Z} a_\gamma^0 e^{2\pi i \gamma \xi}.$$

Similarly, we define

$$A_1(\xi) = \frac{1}{2}\sum_{\gamma \in Z} a_\gamma^1 e^{2\pi i \gamma \xi}.$$

In terms of A_0 and A_1, equations (1.7) and (1.8) can be shown to be equivalent to

$$\sum_{p=1}^{2} A_k(\xi + \eta_p)A_j(\xi + \eta_p) = \delta(j,k), \tag{1.9}$$

and

$$A_0(0) = 1, \tag{1.10}$$

respectively [1] where $\eta_1 = 0$ and $\eta_2 = \frac{1}{2}$. One method of construction presented by Daubechies is to first solve for $\{a_\gamma^0\}$ using

$$A_0(0) = 1$$
$$|A_0(\xi)|^2 + |A_0(\xi + \frac{1}{2})|^2 = 1$$
$$A_0(\xi) \neq 0 \qquad \text{for } |\xi| \leq \tfrac{1}{4},$$

and subsequently incorporate the remaining equations from (1.9) to find $A_1(\xi)$ which in turn gives $\{a_\gamma^1\}$. The remaining equations are

$$|A_1(\xi)|^2 + |A_1(\xi + \frac{1}{2})|^2 = 1, \tag{1.11}$$

and

$$A_0(\xi)A_1(\xi) + A_0(\xi + \frac{1}{2})A_1(\xi + \frac{1}{2}) = 0. \tag{1.12}$$

It is shown [1] that

$$A_1(\xi) = e^{2\pi i \xi}A_0(\xi + \frac{1}{2})$$

solves (1.11) and (1.12) and this amounts to setting

$$a_\gamma^1 = (-1)^{\gamma+1}a_{1-\gamma}^0. \tag{1.13}$$

2 A Construction of Scaling Functions and Their Wavelets

In this section, we present a method of constructing a class of scaling functions from four coefficient scaling filters using only elementary algebraic operations. Particularly, we are interested in obtaining the solutions $a_\gamma \equiv a^0_\gamma$ of equations (1.2), (1.3) and (1.4). Recall from the previous section, each solution will generate a multiresolution analysis provided that condition (1.5) holds. Formulas for constructing four coefficient scaling filters already exist. An elegant approach of Daubechie is well documented in [1]. Therefore, we do not claim that the results obtained in this section are new. But rather, we hope to shed another perspective in the construction using elementary algebraic operations. Once a^0_γ are constructed, the associated wavelets can be generated from (1.13).

We consider the case where only a_0, a_1, a_2 and a_3 are the terms which could possibly be nonzero. We also assume that they are real. Equations (1.2), (1.4) and (1.3) become respectively,

$$\begin{aligned} a_0^2 + a_1^2 + a_2^2 + a_3^2 &= 2 \\ a_0 + a_1 + a_2 + a_3 &= 2 \\ a_0 a_2 + a_1 a_3 &= 0. \end{aligned} \tag{2.1}$$

If $a_3 \neq 0$, then $a_1 = -\frac{a_0 a_2}{a_3}$, and substituting, we get

$$\begin{aligned} a_0^2 + \tfrac{a_0^2 a_2^2}{a_3^2} + a_2^2 + a_3^2 &= 2 \\ a_0 - \tfrac{a_0 a_2}{a_3} + a_2 + a_3 &= 2. \end{aligned}$$

Let $x \equiv a_0$, $y \equiv a_2$ and $c \equiv \frac{1}{a_3}$, thereby obtaining

$$\begin{aligned} x^2 + c^2x^2y^2 + y^2 &= 2 - \tfrac{1}{c^2} \\ x - cxy + y &= 2 - \tfrac{1}{c}. \end{aligned} \tag{2.2}$$

Assuming $1 - cx \neq 0$, (2.2) yields

$$y = \frac{2 - \frac{1}{c} - x}{1 - cx} \quad \text{and} \quad y^2 = \frac{2 - \frac{1}{c^2} - x^2}{1 + c^2x^2}. \tag{2.3}$$

Squaring the first equation and equating it to the second equation yields

$$2c^2x^4 - 4c^2x^3 + (2c^2 - 4c + 4)x^2 + 4(c-1)x + 2 - \frac{4}{c} + \frac{2}{c^2} = 0. \tag{2.4}$$

In order to study the solutions of (2.4), let

$$f(x) = 2c^2x^4 - 4c^2x^3 + (2c^2 - 4c + 4)x^2 + 4(c-1)x + 2 - \frac{4}{c} + \frac{2}{c^2}.$$

First, we observe that, for every $c \neq 0$,

$$f'(\frac{1}{2}) = 0, \tag{2.5}$$

and that

$$f(x + \frac{1}{2}) = 2c^2x^4 + (4 - 4c - c^2)x^2 + \frac{1}{8}c^2 + c + 1 - \frac{4}{c} + \frac{2}{c^2}. \tag{2.6}$$

Equation (2.6) shows that the graph of f is symmetric about the line $x = \frac{1}{2}$. Let $\tilde{f}(x) \equiv f(x+\frac{1}{2})$. Then $\tilde{f}'(x) = 0$ has roots

$$0 \quad \text{and} \quad \pm\frac{1}{2}\sqrt{1 + \frac{4}{c} - \frac{4}{c^2}}.$$

When $\pm\frac{1}{2}\sqrt{1 + \frac{4}{c} - \frac{4}{c^2}}$ are real, -i.e., for

$$c < -2 - 2\sqrt{2} \quad \text{or} \quad c > -2 + 2\sqrt{2}, \tag{2.7}$$

$\tilde{f}$ attains its minimum at $\pm\frac{1}{2}\sqrt{1 + \frac{4}{c} - \frac{4}{c^2}}$. For the values of c specified in (2.7), we have $\tilde{f}(\pm\frac{1}{2}\sqrt{1 + \frac{4}{c} - \frac{4}{c^2}}) = 0$. Translating back into f, we conclude that

$$\frac{1}{2} \pm \frac{1}{2}\sqrt{1 + \frac{4}{c} - \frac{4}{c^2}}, \tag{2.8}$$

are the roots of $f(x) = 0$. Each is a double root of the quartic $f(x) = 0$. If we denote two roots in (2.8) by x_1 and x_2, then notice that $x_2 = 1 - x_1$. This, of course, is a rehash of the fact that the graph of f is symmetric about $x = \frac{1}{2}$. The following theorem serves as the first step toward establishing the algorithm that generates many wavelets.

Theorem 2.1 *Consider the equations in (2.1) and let $x \equiv a_0$, $y \equiv a_2$ and $c \equiv \frac{1}{a_3}$. Then $x + y = 1$.*

Proof: Since $x_1 = \frac{1}{2} + \frac{1}{2}\sqrt{1 + \frac{4}{c} - \frac{4}{c^2}}$ and $x_2 = \frac{1}{2} - \frac{1}{2}\sqrt{1 + \frac{4}{c} - \frac{4}{c^2}}$ are two roots of $f(x) = 0$,

$$(x - \frac{1}{2} - \frac{1}{2}\sqrt{1 - \frac{4}{c^2} + \frac{4}{c}})(x - \frac{1}{2} - \frac{1}{2}\sqrt{1 - \frac{4}{c^2} + \frac{4}{c}}),$$

are two linear factors of $f(x)$, and upon multiplying and simplifying, we obtain $x^2 - x + \frac{1}{c^2} - \frac{1}{c}$. If $x = x_1$ or $x = x_2$, then $x^2 - x + \frac{1}{c^2} - \frac{1}{c} = 0$, which reduces

$$cx - 1 = cx^2 + \frac{1}{c} - 2. \tag{2.9}$$

But we have, from (2.3) and (2.9),

$$x + y = x + \frac{2 - \frac{1}{c} - x}{1 - cx} = \frac{-cx^2 - \frac{1}{c} + 2}{1 - cx} = 1.$$

$\square$

Note that from (2.1) and Theorem 2.1, we obtain

$$a_0 + a_2 = 1 \quad \text{and} \quad a_1 + a_3 = 1. \tag{2.10}$$

One might argue that (2.10) follows from (1.13) as $A_1(0) = 0$, which is the well known vanishing moment condition for wavelets. This is true, of course, but adding the condition (2.10) to (2.1) to solve for the parameters for the scaling function defeats the main purpose of this note. The purpose here is to give a solution method to (2.1)-Step 1 of Section 1- as well as to derive also the vanishing moment property by using only elementary algebraic means. In other words, an alternative approach was given to the method of Daubechie which is described in (1.9)-(1.12). In [2] (p. 296), it is noted that "the sum condition $\sum a_{even} = \sum a_{odd} = 1$ is always imposed". This condition is taken primarily because it is a special case of the following condition for accuracy of wavelets of order p:

$$\sum (-1)^k k^m a_k = 0, \qquad \text{for } m < p. \tag{2.11}$$

More specifically, with $p = 1$, equation (2.11) is $a_0 - a_1 + a_2 - a_3 = 0$, which is satisfied by $\sum a_{even} = \sum a_{odd} = 1$. Equations in (2.10) tell us that, with wavelets whose scaling relation involves four terms, it is necessary that $\sum a_{even} = \sum a_{odd} = 1$. Some immediate consequences of Theorem 2.1 are:

$$\begin{aligned}
(a_0 - \tfrac{1}{2})^2 + (a_1 - \tfrac{1}{2})^2 &= \tfrac{1}{2} \\
(a_2 - \tfrac{1}{2})^2 + (a_2 - \tfrac{1}{2})^2 &= \tfrac{1}{2} \\
(a_0 - \tfrac{1}{2})^2 + (a_3 - \tfrac{1}{2})^2 &= \tfrac{1}{2} \\
(a_2 - \tfrac{1}{2})^2 + (a_3 - \tfrac{1}{2})^2 &= \tfrac{1}{2} \\
a_1^2 - a_1 - a_0 a_2 &= 0 \\
a_3^2 - a_3 - a_0 a_2 &= 0.
\end{aligned}$$

Now, we are ready to present the algorithm for the solution of our problem:

ALGORITHM:

1. Choose $a_3 \in (\frac{1}{-2-2\sqrt{2}}, \frac{1}{-2+2\sqrt{2}})$.

2. Find $a_0 = \frac{1}{2} \pm \frac{1}{2}\sqrt{1 + 4a_3 - 4a_3^2}$.

3. Get $a_2 = 1 - a_0$.

4. Get $a_1 = 1 - a_3$.

The corresponding scaling function φ can be derived by finding a fixed point of the mapping

$$Sf(x) = a_0 f(2x) + a_1 f(2x-1) + a_2 f(2x-2) + a_3 f(2x-3),$$

by iteration with a reasonable starting function, f^0, -i.e.,

$$\varphi(x) = \lim_{n\to\infty} S^n f^0(x).$$

By (2.10), (2.11) is satisfied with $m = 0$. Therefore, the algorithm, when a_3 is taken in the range specified, will generates a family of wavelets whose accuracy order is 1. If we want the second order accuracy in wavelets, then we must have equation (2.11) with $m = 1$. In our case, this is

$$-a_1 + 2a_2 - 3a_3 = 0. \tag{2.10}$$

Since $a_1 = 1 - a_3$ and $a_0 = 1 - a_2$, substituting into (2.10), we obtain $2a_0 = 1 - 2a_3$. Replacing a_0 by $\frac{1}{2} \pm \frac{1}{2}\sqrt{1 + 4a_3 - 4a_3^2}$ and solving for a_3, we obtain $a_3 = \frac{1\pm\sqrt{3}}{4}$. If we take $a_3 = \frac{1-\sqrt{3}}{4}$, then the following wavelet is generated by the algorithm;

$$a_0 = \frac{1+\sqrt{3}}{4}, \quad a_1 = \frac{3+\sqrt{3}}{4}, \quad a_2 = \frac{3-\sqrt{3}}{4}, \quad a_3 = \frac{1-\sqrt{3}}{4}.$$

Of course, this is one of Daubechies wavelets. It is important to note that our approach can be extended to any scaling function that involves four nonzero coefficients. For instance, consider the case;

$$\varphi(x) = a_{-1}\varphi(2x+1) + a_0\varphi(2x) + a_1\varphi(2x-1) + a_2\varphi(2x-2).$$

Then the algorithm is modified to

1. Choose $a_2 \in (\frac{1}{-2-2\sqrt{2}}, \frac{1}{-2+2\sqrt{2}})$.

2. Find $a_{-1} = \frac{1}{2} \pm \frac{1}{2}\sqrt{1 + 4a_2 - 4a_2^2}$.

3. Get $a_1 = 1 - a_{-1}$.

4. Get $a_0 = 1 - a_2$.

Arguing as before to obtain the second order accuracy in the wavelet, we get once again the coefficients,

$$a_{-1} = \frac{1+\sqrt{3}}{4}, \quad a_0 = \frac{3+\sqrt{3}}{4}, \quad a_1 = \frac{3-\sqrt{3}}{4}, \quad a_2 = \frac{1-\sqrt{3}}{4}.$$

The support of the scaling function is $[-1, 2]$. Also, we note that choosing $a_2 = 0$ and the minus sign for a_{-1} in the algorithm, yields $a_{-1} = a_2 = 0$, $a_0 = a_1 = 1$. These values yield the scaling function φ for the classical Haar wavelets.

References

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