Solutions for MATH 212 Test 1 Spring 2001

1. (70pts) Evaluate the following integrals.

a. $\int \frac{x^2}{(x^3+6)^{3/2}}\,dx$

Let $u = x^3 + 6$ so that $du = 3x^2 dx$. Then

$$\begin{aligned}\int \frac{x^2}{(x^3+6)^{3/2}}\,dx &= \frac{1}{3}\int \frac{du}{u^{3/2}} = -\frac{2}{3}u^{-1/2} + C\\ &= -\frac{2}{3}(x^3+6)^{-1/2} + C.\end{aligned}$$

b. $\int (x+7)\ln x\,dx$

Let $u = \ln x$ and $dv = (x+7)dx$ so that $du = \frac{1}{x}dx$ and $v = \frac{1}{2}x^2 + 7x$. Using integration by parts,

$$\begin{aligned}\int (x+7)\ln x\,dx &= (\frac{1}{2}x^2 + 7x)\ln x - \int (\frac{1}{2}x^2 + 7x)\frac{1}{x}\,dx\\ &= (\frac{1}{2}x^2 + 7x)\ln x - \int (\frac{1}{2}x + 7)\,dx\\ &= (\frac{1}{2}x^2 + 7x)\ln x - \frac{1}{4}x^2 - 7x + C.\end{aligned}$$

c. $\int \tan^3 x \sec^4 x\,dx$

Let $u = \sec x$ so that $du = \sec x \tan x\,dx$. Then

$$\begin{aligned}\int \tan^3 x\sec^4 x\,dx &= \int \tan^2 x \sec^3 x \sec x \tan x\,dx\\ &= \int (\sec^2 x - 1)\sec^3 x \sec x \tan x\,dx\\ &= \int (u^2-1)u^3\,du = \int \frac{1}{6}\sec^6 x - \frac{1}{4}\sec^4 x + C.\end{aligned}$$

d. $\int \frac{x^2}{(9-x^2)^{3/2}}\,dx$

Let $x = 3\sin\theta$ so that $dx = 3\cos\theta d\theta$. Then

$$\begin{aligned}\int \frac{x^2}{(9-x^2)^{3/2}}\,dx &= \int \frac{9\sin^2\theta}{27\cos^3\theta}3\cos\theta d\theta\\ &= \int \frac{\sin^2\theta}{\cos^2\theta}d\theta = \int \tan^2\theta d\theta\\ &= \int \sec^2\theta - 1\,d\theta = \tan\theta - \theta + C\\ &= \frac{x}{\sqrt{9-x^2}} - \sin^{-1}(\frac{x}{3}) + C.\end{aligned}$$

e. $\int \cos x \ln^2(\sin x)\, dx$

Let $t = \sin x$ so that $dt = \cos x dx$. Then

$$\int \cos x \ln^2(\sin x)\, dx = \int (\ln t)^2\, dt.$$

Now let $u = (\ln t)^2$ and $dv = dt$ so that $du = \frac{2\ln t}{t}$ and $v = t$.

$$\begin{aligned}\int (\ln t)^2\, dt &= t(\ln t)^2 - 2\int \ln t dt \\ &= t(\ln t)^2 - 2(t\ln t - t) + C \\ &= \sin x(\ln \sin x)^2 - 2(\sin x(\ln \sin x) - \sin x) + C.\end{aligned}$$

f. $\int \frac{x}{x^2-2x+3} dx$

$$\begin{aligned}\int \frac{x}{x^2-2x+3} dx &= \frac{x}{(x-1)^2+2}\, dx \\ &= \int \frac{\sqrt{2}\tan\theta + 1}{2\sec^2\theta}\sqrt{2}\sec^2\theta d\theta \\ &\quad \text{here we let } x-1 = \sqrt{2}\tan\theta \\ &= \frac{1}{\sqrt{2}}\int \sqrt{2}\tan\theta + 1 \\ &= \frac{1}{\sqrt{2}}(\ln\theta + \theta) + C \\ &= \ln|\frac{\sqrt{(x-1)^2+2}}{\sqrt{2}}| + \tan^{-1}(\frac{x-7}{\sqrt{2}}) + C.\end{aligned}$$

g. $\int \frac{x^2}{(x+1)^3} dx$

Let $u = x+1$ so that $du = dx$. Then

$$\begin{aligned}\int \frac{x^2}{(x+1)^3} dx &= \int \frac{(u-1)^2}{u^3} du \\ &\int \frac{1}{u} - \frac{2}{u^2} + \frac{1}{u^3} du \\ &= \ln u + \frac{1}{u} - \frac{1}{2u^2} + C \\ &= \ln|x+1| + \frac{1}{x+1} - \frac{1}{2(x+1)^2} + C.\end{aligned}$$

h. $\int_1^{\infty} \frac{1}{x^2+1}\, dx$

$$\begin{aligned}\int_1^{\infty} \frac{1}{x^2+1}\, dx &= \lim_{b\to\infty} \int_1^{b} \frac{1}{x^2+1} dx \\ &= \lim_{b\to\infty} [\arctan b - \arctan 1] = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}.\end{aligned}$$

i. $\int_0^{1} \frac{1}{x^2-1}\, dx$

By partial fractions,

$$\begin{aligned}\int_0^{1} \frac{1}{x^2-1} &= \int_0^{1} (\frac{1/2}{x+1} - \frac{1/2}{x-1}) dx \\ &= \frac{1}{2} \ln|x+1|\,|_0^1 - \lim_{\epsilon\to 0^+} \frac{1}{2} \ln|x-1|\,|_0^{1-\epsilon} \\ &= \infty.\end{aligned}$$

j. $\int x^5 e^{x^2}\, dx$

Let $t = x^2$ so that $dt = 2xdx$ and

$$\begin{aligned}\int x^5 e^{x^2}\, dx &= \frac{1}{2}\int t^2 e^t\, dt \\ &= t^2e^t - 2\int te^t\, dt \quad (u = t^2,\ dv = e^t\, dt) \\ &= t^2e^t - 2[te^t - e^t] + C \quad (u = t,\ dv = e^t\, dt) \\ &= x^4 e^{x^2} - 2x^2 e^{x^2} + 2e^{x^2} + C\end{aligned}$$

2. (15pts) Find the volume of the solid obtained if the region bounded by $y = x$ and $y = 4x - x^2$ in the first quadrant is rotated about (a) the x-axis and (b) the y axis.

(a) The graphs of $y = x$ and $y = 4x - x^2$ intersect at 0 and 3, (solve $x = 4x - x^2$ for x). Hence using the solids of revolution formula,

$$V = \int_0^3 \pi[(4x - x^2)^2 - x^2]dx = \frac{108\pi}{5}$$

(b) Using the cylindrical shell method,

$$V = \int_0^3 2\pi[(4x - x^2) - x]dx = \frac{27\pi}{2}$$

3. (15pts) Consider the region in the first quadrant that is bounded by $y = x^2$ and $y = x^3$. Set up the integrals (do not evaluate) that compute the volume of the solid obtained when the region is rotated about

a. the line $y = -1$ The graphs of $y = x^2$ and $y = x^3$ intersect at $x = 0$ and $x = 1$.

$$V = \int_0^1 \pi[(x^2 + 1)^2 - (x^3 + 1)^2]dx$$

b. the line $x = 3$.

$$V = \int_0^1 2\pi(3 - x)[x^2 - x^3]dx.$$