MATH 212-Solutions   Test 2   Spring 2001

Each problem is worth 10 points. You must show your work to receive credits.

1. Sketch the curve given by $x = -1 + \cos t$ and $y = 2 + \sin t$, $0 \le t \le 2\pi$. Indicate with an arrow the motion of a particle.

   • The curve is the circle $(x+1)^2+(y-2)^2 = 1$ and the orientation is counter-clockwise.

2. Given $x = 1 + t^2$ and $y = t \ln t$, find dy/dx and d^2y/dx^2.

   • $\frac{dy}{dx} = \frac{\ln t+1}{2t}$, and $\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}(\frac{\ln t+1}{2t})}{\frac{dx}{dt}} = \frac{-\ln t}{4t^3}$.

3. Find equations of the tangents to the curve $x = 3t^2 + 1$, $y = 2t^3 + 1$ that pass through the point $(4, 3)$.

   • Set $x = 4$ and $y = 3$ to find $t = 1$ at which time a particle is at $(4, 3)$. $\frac{dy}{dx} = \frac{6t^2}{6t}$ which is 1 when $t = 1$. Hence an equation of the tangent is $y = x - 1$.

4. Find the length of the curve $x = e^t + e^{-t}$, $y = 5 - 2t$, $0 \le t \le 3$.

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\begin{aligned}
L &= \int_0^3 \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2}dt \\
&= \int_0^3 \sqrt{(e^t - e^{-t})^2 + (-2)^2}dt \\
&= \int_0^3 \sqrt{(e^t + e^{-t})^2}dt \\
&= e^3 - e^{-3}.
\end{aligned}
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5.

   a. Find a Cartesian equation for the curve $r = \frac{1}{1+2\sin\theta}$.

   • $r = \frac{1}{1+2\sin\theta} \Rightarrow r + 2r\sin\theta = 1$, which reduces to $x^2 + y^2 = (1 - 2y)^2$.

   b. Sketch $r = 2\cos 3\theta$.

   Rose with 3 petals.

6. Sketch $r = 1 - \sin\theta$, and find the points on the curve where the tangent line is horizontal. Recall that $\frac{dy}{dx} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta}$.

The graph is a cardioid. As $\frac{dr}{d\theta} = -\cos\theta$, $\frac{dr}{d\theta}\sin\theta + r\cos\theta = \cos\theta[1 - 2\sin\theta]$ When this is set zero, we obtain $\theta = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{\pi}{6}, \frac{5\pi}{6}$. Not that the denominator $\frac{dr}{d\theta}\cos\theta - r\sin\theta$ is zero for $\theta = \frac{\pi}{2}$, so the horizontal tangents occur at $\theta = \frac{3\pi}{2}, \frac{\pi}{6}, \frac{5\pi}{6}$.

7. Find the area of the region enclosed by one loop of $r = 2\cos 3\theta$. (see 5-b.)

$$\begin{aligned}\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \frac{1}{2}(2\cos 3\theta)^2 d\theta &= 2\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \frac{1+\theta 6\theta}{2} d\theta \\ &= \theta + \frac{\sin 6\theta}{6}\Big|_{-\frac{\pi}{6}}^{\frac{\pi}{6}} = \frac{\pi}{3}.\end{aligned}$$

8. Find the length of the polar curve $r = \theta^2$, $0 \le \theta \le 2\pi$. Sketch the polar curve.

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$$\begin{aligned}\int_0^{2\pi} \sqrt{r^2 + (\frac{dr}{d\theta})^2} d\theta &= \int_0^{2\pi} \theta\sqrt{4+\theta^2} d\theta \\ &= \frac{(4+\theta^2)^{3/2}}{3}\Big|_0^{2\pi} = \frac{1}{3}\{[4+(2\pi)^2]^{3/2} - 8\}.\end{aligned}$$

9. Find the area of the region that lies inside $r = 1 + \cos\theta$ and outside $r = 1$. Sketch the region.

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$$\begin{aligned}Area &= 2[\frac{1}{2}\int_0^{\pi/2} (1+\cos\theta)^2 - 1^2] d\theta \\ &= \int_0^{\pi/2} 2\cos\theta + \cos^2\theta d\theta \\ &= \int_0^{\pi/2} 2\cos\theta + \frac{1+\cos 2\theta}{2} d\theta \\ &= 2 + \frac{\pi}{4}.\end{aligned}$$

10. Find an equation of the ellipse with foci $(3, \pm 1)$ and vertices $(3, \pm 3)$.

$c = 1$, $a = 3$ and $b^2 = 8$ imply

$$\frac{(x-3)^2}{8} + \frac{y^2}{9} = 1.$$