1. (15pts) Determine whether the sequence is convergent or divergent. If it converges, find its limit.
   a. \( a_n = \frac{n}{\ln n} \)
      \[ \lim_{n \to \infty} \frac{n}{\ln n} = \infty \]
   b. \( a_n = \frac{4^n}{5^n} \)
      \[ \lim_{n \to \infty} \frac{4^n}{5^n} = 0 \]
   c. \( a_n = \sqrt[n+3]{x} - \sqrt[n]{x} \)
      \[ \lim_{n \to \infty} \frac{(\sqrt[n+3]{x} - \sqrt[n]{x})(\sqrt[n+3]{x} + \sqrt[n]{x})}{\sqrt[n+3]{x} + \sqrt[n]{x}} = \lim_{n \to \infty} \frac{3}{\sqrt[n+3]{x} + \sqrt[n]{x}} = 0. \]

2. (40pts) Determine whether series is convergent or divergent.
   a. \( \sum_{n=1}^{\infty} \frac{1}{2^n + n} \)
      The series converges by Direct Comparison Test, compared with the convergent series \( \sum_{n=1}^{\infty} \frac{1}{2^n} \), as \( \frac{1}{2^n + n} < \frac{1}{2^n} \).
   b. \( \sum_{n=1}^{\infty} (-1)^n \frac{1}{\sqrt[n+1]{n+1}} \)
      The series converges by Alternating series Test.
   c. \( \sum_{n=2}^{\infty} \frac{1}{n(n \ln n)^2} \)
      The series converges by the integral test, as
      \[ \int_{2}^{\infty} \frac{1}{x(\ln x)^2} dx = \lim_{b \to \infty} \int_{\ln 2}^{b} \frac{1}{u^2} du, \quad u = \ln x \]
      \[ = \lim_{b \to \infty} -\frac{1}{2u^2} \]
      \[ = \frac{1}{2 \ln^2 2} \]
   d. \( \sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{5^n n!} \)
      The series converges by Ratio Test, as \( \lim_{n \to \infty} \left| \frac{1 \cdot 3 \cdot 5 \cdots (2(n+1)-1)}{5^{n+1} (n+1)!} \right| = \frac{2^{n+1}}{5^{n+1} (n+1)!} \) = \( \lim_{n \to \infty} 2^{n+1} \) = \( \frac{2}{5} < \frac{1}{2} \).
   e. \( \sum_{n=1}^{\infty} \frac{n^2 + 1}{5^n n!} \)
      As \( \lim_{n \to \infty} \frac{n^2 + 1}{5^n n!} = \frac{1}{5} \neq 0 \), the series diverges by Test for Divergence.

3. (10pts) Find the sum of the following series.
   a. \( \sum_{n=1}^{\infty} \frac{2^{n+2}}{5^n} \)
   b. \( \sum_{n=1}^{\infty} \frac{1}{n(n+2)} \)
      The series converges by Comparison Test, compared with \( \sum_{n=1}^{\infty} \frac{1}{n^2} \).
      \[ \sum_{n=1}^{\infty} \frac{27 \cdot 3^{n-1}}{5 \cdot 3^{n-1}} = \sum_{n=1}^{\infty} \frac{27}{5} = \frac{27}{5} \]
      \[ \sum_{n=1}^{\infty} \frac{1}{n(n+2)} = \frac{27}{5} \]
\[
\sum_{n=1}^{\infty} \frac{1}{n(n+2)} = \sum_{n=1}^{\infty} \frac{1}{2} \left( \frac{1}{n} - \frac{1}{n+2} \right)
= \frac{1}{2} \left\{ (1 - \frac{1}{3}) + (\frac{1}{2} - \frac{1}{4}) + (\frac{1}{3} - \frac{1}{5}) + \cdots \right\}
= \frac{1}{2} + \frac{1}{2} = \frac{3}{2}.
\]

4.(15pts) Determine whether the series is absolutely convergent, conditionally convergent or divergent.

a. \[\sum_{n=1}^{\infty} \frac{(-1)^n}{\ln n}\]
   - The series converges conditionally, as \[\sum_{n=1}^{\infty} \frac{(-1)^n}{\ln n}\] converges by Alt. Series Test, whereas, \[\sum_{n=1}^{\infty} \left| \frac{(-1)^n}{\ln n} \right| = \sum_{n=1}^{\infty} \frac{1}{\ln n}\] diverges by Direct Comp. Test with \[\sum_{n=1}^{\infty} \frac{1}{n}\].

b. \[\sum_{n=1}^{\infty} \frac{(-1)^n}{n^4}\]
   - The series converges absolutely.

c. \[\sum_{n=1}^{\infty} \frac{(-1)^n}{n^e - n}n!\]
   - The series converges by Ratio Test as
     \[
     \lim_{n \to \infty} \left| \frac{(-1)^{n+1} e^{-n}n!}{(-1)^n} \right| = \lim_{n \to \infty} \frac{e}{n+1} = 0
     \]

5.(10pts) Find the radius of convergence and interval of convergence of \[\sum_{n=1}^{\infty} \frac{(-1)^n(x+2)^n}{n2^n}\].

\[
\lim_{n \to \infty} \left| \frac{(x+2)^{n+1}}{(n+1)^{2^{n+1}}} \cdot \frac{n2^n}{(x+2)^n} \right| = \frac{|x+2|}{2}.
\]
By Ratio Test, the power series converges for those \(x\) for which \[\frac{|x+2|}{2} < 1\], or \(-4 < x < 0\).
Now we check the end points for convergence. For \(x = -4\),
\[
\sum_{n=1}^{\infty} (-1)^n \frac{(x+2)^n}{n2^n} = \sum_{n=1}^{\infty} (-1)^n \frac{(-2)^n}{n2^n} = \sum_{n=1}^{\infty} \frac{1}{n}
\]
which is the divergent Harmonic series. For \(x = 0\),
\[
\sum_{n=1}^{\infty} (-1)^n \frac{(x+2)^n}{n2^n} = \sum_{n=1}^{\infty} (-1)^n \frac{1}{n}
\]
which is convergent by the Alt. series Test. Hence \(R = 2\) and Interval\((-4,0]\).

6.(10pts) Express \(\frac{1}{1+x^2}\) as a power series and find the interval of convergence. Use the result obtained, find the Maclaurin series for \(\tan^{-1} x\).
\[ \frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty}(-x^2)^n = \sum_{n=0}^{\infty}(-1)^n x^{2n} \] where the convergence takes place for \(|-x^2| < 1\) or \(-1 < x < 1\). Within this interval of convergence,

\[ \tan^{-1} x = \int_0^x \frac{1}{1+t^2} \, dt = \int_0^x \sum_{n=0}^{\infty} (-1)^n t^{2n} \, dt = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{t^{2n-1}}{(2n-1)!}. \]

7. Find the Taylor series for \( f(x) = \cos x \) centered at \( \pi/4 \).

- As \( f'(x) = -\sin x, f''(x) = -\cos x, f'''(x) = \sin x, f^{(4)}(x) = \cos x \), etc. \( f(\pi/4) = \frac{\sqrt{2}}{2}, f'(\pi/4) = -\frac{\sqrt{2}}{2}, f''(\pi/4) = -\frac{\sqrt{2}}{2}, f'''(\pi/4) = \frac{\sqrt{2}}{2} \) and \( f^{(4)}(\pi/4) = \frac{\sqrt{2}}{2} \). Hence

\[ \cos x = \sum_{n=0}^{\infty} (-1)^{n+1} \frac{\sqrt{2}}{2 n!} (x - \frac{\pi}{4})^n. \]