MATH 212   Test 3   Spring 2001

You must show your work to get credit.

1.(15pts) Determine whether the sequence is convergent or divergent. If it converges, find its limit.

a. $a_n = \frac{n}{\ln n}$

• $\lim_{n\to\infty} \frac{n}{\ln n} = \infty$

b. $a_n = \frac{4^{n+10}}{5^n}$

• $\lim_{n\to\infty} \frac{4^{n+10}}{5^n} = 0$

c. $a_n = \sqrt{x+3} - \sqrt{x}$

• $\lim_{n\to\infty} \frac{(\sqrt{x+3}-\sqrt{x})(\sqrt{x+3}+\sqrt{x})}{\sqrt{x+3}+\sqrt{x}} = \lim_{n\to\infty} \frac{3}{\sqrt{x+3}+\sqrt{x}} = 0.$

2.(40pts) Determine whether series is convergent or divergent.

a. $\sum_{n=1}^{\infty} \frac{1}{2^n+n}$

• The series converges by Direct Comparison Test, compared with the convergent series $\sum_{n=1}^{\infty} \frac{1}{2^n}$, as $\frac{1}{2^n+n} < \frac{1}{2^n}$.

b. $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt[3]{n+1}}$

• The series converges by Alternating series Test.

c. $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^3}$

• The series converges by the integral test, as

$$\begin{aligned}\int_2^{\infty} \frac{1}{x(\ln x)^3}dx &= \lim_{b\to\infty} \int_{\ln 2}^{b} \frac{1}{u^3}du,\ u = \ln x \\ &= \lim_{b\to\infty} -\frac{1}{2u^2} \\ &= \frac{1}{2\ln^2 2}\end{aligned}$$

d. $\sum_{n=1}^{\infty} \frac{1\cdot 3\cdot 5\cdots(2n-1)}{5^n n!}$

• The series converges by Ratio Test, as $\lim_{n\to\infty} |\frac{1\cdot 3\cdot 5\cdots(2(n+1)-1)}{5^{n+1}(n+1)!}\frac{5^n n!}{1\cdot 3\cdot 5\cdots(2n-1)}| = \lim_{n\to\infty} \frac{2n+1}{5(n+1)} = \frac{2}{5} < 1.$

e. $\sum_{n=1}^{\infty} \frac{n^2+1}{5n^2+n}$

• As $\lim_{n\to\infty} \frac{n^2+1}{5n^2+n} = \frac{1}{5} \neq o$, the series diverges by Test for Divergence.

3.(10pts) Find the sum of the following series.

a. $\sum_{n=1}^{\infty} \frac{3^{n+2}}{5^n}$

• $\sum_{n=1}^{\infty} \frac{3^{n+2}}{5^n} = \sum_{n=1}^{\infty} \frac{27\cdot 3^{n-1}}{5\cdot 5^{n-1}} = \frac{27/5}{1-3/5} = \frac{27}{2}.$

b. $\sum_{n=1}^{\infty} \frac{1}{n(n+2)}$

- $$\begin{aligned}\sum_{n=1}^{\infty}\frac{1}{n(n+2)} &= \sum_{n=1}^{\infty}\frac{1}{2}\{\frac{1}{n}-\frac{1}{n+2}\} \\ &= \frac{1}{2}\{(1-\frac{1}{3})+(\frac{1}{2}-\frac{1}{4})+(\frac{1}{3}-\frac{1}{5})+\cdots\} \\ &= 1+\frac{1}{2}=\frac{3}{2}.\end{aligned}$$

4.(15pts) Determine whether the series is absolutely convergent, conditionally convergent or divergent.

a. $\sum_{n=1}^{\infty}\frac{(-1)^n}{\ln n}$

• The series converges conditionally, as $\sum_{n=1}^{\infty}\frac{(-1)^n}{\ln n}$ converges by Alt. Series Test, whereas, $\sum_{n=1}^{\infty}|\frac{(-1)^n}{\ln n}|=\sum_{n=1}^{\infty}\frac{1}{\ln n}$ diverges by Direct Comp. Test with $\sum_{n=1}^{\infty}\frac{1}{n}$.

b. $\sum_{n=1}^{\infty}\frac{(-1)^n}{4^n}$

• The series converges absolutely.

c. $\sum_{n=1}^{\infty}\frac{(-1)^n}{e^{-n}n!}$

• The series converges by Ratio Test as

$$\lim_{n\to\infty}|\frac{(-1)^{n+1}}{e^{-(n+1)}(n+1)!}\frac{e^{-n}n!}{(-1)^n}|=\lim_{n\to\infty}\frac{e}{n+1}=0$$

5.(10pts) Find the radius of convergence and interval of convergence of $\sum_{n=1}^{\infty}(-1)^n\frac{(x+2)^n}{n2^n}$.

- $$\lim_{n\to\infty}|\frac{(x+2)^{n+1}}{(n+1)2^{n+1}}\cdot\frac{n2^n}{(x+2)^n}|=\frac{|x+2|}{2}.$$

By Ratio Test, the power series converges for those x for which $\frac{|x+2|}{2}<1$, or $-4<x<0$. Now we check the end points for convergence. For $x=-4$,

$$\sum_{n=1}^{\infty}(-1)^n\frac{(x+2)^n}{n2^n}=\sum_{n=1}^{\infty}(-1)^n\frac{(-2)^n}{n2^n}=\sum_{n=1}^{\infty}\frac{1}{n}$$

which is the divergent Harmonic series. For $x=0$,

$$\sum_{n=1}^{\infty}(-1)^n\frac{(x+2)^n}{n2^n}=\sum_{n=1}^{\infty}(-1)^n\frac{1}{n}$$

which is convergent by the Alt. series Test. Hence $R=2$ and Interval-$(-4,0]$.

6.(10pts) Express $\frac{1}{1+x^2}$ as a power series and find the interval of convergence. Use the result obtained, find the Maclaurin series for $\tan^{-1}x$.

- $\frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty}(-x^2)^n = \sum_{n=0}^{\infty}(-1)^n x^{2n}$ where the convergence takes place for $|-x^2| < 1$ or $-1 < x < 1$. Within this interval of convergence,

$$\tan^{-1} x = \int_0^x \frac{1}{1+t^2} = \int_0^x \sum_{n=0}^{\infty}(-1)^n t^{2n} dt = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{t^{2n-1}}{(2n-1)!}.$$

7. Find the Talyor series for $f(x) = \cos x$ centered at $\pi/4$.

- As $f'(x) = -\sin x$, $f''(x) = -\cos x$, $f'''(x) = \sin x$, $f^{(4)}(x) = \cos x$, etc. $f(\pi/4) = \frac{\sqrt{2}}{2}$, $f'(\pi/4) = -\frac{\sqrt{2}}{2}$, $f''(\pi/4) = -\frac{\sqrt{2}}{2}$, $f'''(\pi/4) = \frac{\sqrt{2}}{2}$ and $f^{(4)}(\pi/4) = \frac{\sqrt{2}}{2}$. Hence

$$\cos x = \sum_{n=0}^{\infty}(-1)^{\frac{n(n+1)}{2}}\frac{\sqrt{2}}{2}\frac{1}{n!}(x-\frac{\pi}{4})^n.$$