MATH 316
Introductory Linear Algebra
Fall 2001

Sample Final Exam

1. Find all vectors in $\mathbb{R}^4$ that are perpendicular to the three vectors

$$\begin{pmatrix} 1 \\ 2 \\ -1 \\ 0 \end{pmatrix} \quad \begin{pmatrix} 0 \\ 4 \\ 4 \\ -1 \end{pmatrix} \quad \begin{pmatrix} 2 \\ 0 \\ -6 \\ 1 \end{pmatrix}.$$

2. Consider the homogeneous linear system $A\vec{x} = \vec{0}$. Justify the following facts:

   (a) All homogeneous systems are consistent.

   (b) A homogeneous system with fewer equations than unknowns has infinitely many solutions.

3. Define $A = \begin{pmatrix} 1 & -b \\ b & 1 \end{pmatrix}$ and $T(\vec{x}) = A\vec{x}$.

   (a) For which choice of the constant b is the matrix A invertible?

   (b) What is the inverse in this case?

   (c) Give a geometric interpretation of the linear transformation for $b = 1$.

4. Let L be the line in $\mathbb{R}^3$ that consists of all scalar multiples of the vector $\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$. Find the orthogonal projection of the vector $\begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$ onto L.

5. Find the inverse of $A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 3 & 2 \\ 3 & 8 & 2 \end{pmatrix}$.

6. Let $W = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} : x + 2y = 0,\ z = 0 \right\}$.

   (a) Show that W is a subspace of $\mathbb{R}^3$.

   (b) Find a basis for W.

   (c) What is the dimension of W?

7. Consider the matrix $A = \begin{pmatrix} 1 & 1 & 2 & 2 \\ 1 & 2 & 2 & 3 \\ 1 & 3 & 2 & 4 \end{pmatrix}$. Find a basis for $\ker(A)$ and a basis for $\operatorname{im}(A)$.

8. Which of the subsets of $\mathbb{R}^{3\times 3}$ given below are subspaces of $\mathbb{R}^{3\times 3}$? Argue carefully. In a case of a subspace, specify its dimension by exhibiting a basis.

   (a) The invertible 3×3 matrices.

   (b) The upper triangular 3×3 matrices.

   (c) The 3×3 matrices A such that the vector $\begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$ belongs to $\ker(T)$ where $T(\vec{x}) = A\vec{x}$.

9. Consider the basis $\mathcal{B} = \{1-t, 1+t, t^2\}$ for $\mathbb{P}^2$, the space of all polynomials of degree ≤ 2. Let $T\colon \mathbb{P}^2 \to \mathbb{P}^2$ be a linear transformation whose $\mathcal{B}-$matrix is given by

$$B_T = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 5 & 2 \\ 0 & 0 & 3 \end{pmatrix}.$$

   (a) Is T an isomorphism?

   (b) Find $T(1 + 2t + 3t^2)$.

10. Consider the matrix $B = \begin{pmatrix} 1 & 1 & 1 \\ a & b & t \\ a^2 & b^2 & t^2 \end{pmatrix}$.

(a) Express $\det(B)$ as a function of t.

(b) For which values of t is the matrix B invertible?

11. Let $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$.

(a) Find the characteristic polynomial of A.

(b) Find the eigenvalues of A and their algebraic multiplicities.

(c) Find the associated eigenvectors and the geometric multiplicities of the eigenvalues.

(d) Find an invertible matrix S such that

$$AS = S \begin{pmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

12. Let

$$A = \begin{pmatrix} 1 & 6 & 1 & 1 \\ 2 & 1 & 2 & 1 \\ 0 & 5 & 0 & 0 \\ -1 & 2 & 3 & 0 \end{pmatrix}$$

Compute $\det(A), \det(A^{-1}), \det(2A)$, $\text{rank}(A)$ and the nullity of A.

13. Consider the following sets of vectors in $\mathbb{R}^3$:

$$S_1 = \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \right\}, \quad S_2 = \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \right\},$$

$$S_3 = \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\}, \quad S_4 = \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\},$$

$$S_5 = \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix} \right\}, \quad S_6 = \left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\}.$$

Which of these sets are linearly independent? Which of them span $\mathbb{R}^3$? Which of them form bases for $\mathbb{R}^3$? Which of them span subspaces of $\mathbb{R}^3$?

14. Suppose $\vec{v}$ is an eigenvector of $A \in \mathbb{R}^n$ with associated eigenvalue 2.

    (a) Argue why $\vec{v}$ is an eigenvector of $B = A^2 - 3A + 2I_n$. What is the associated eigenvalue?

    (b) Can B be invertible?

15. Suppose $A = \begin{pmatrix} 1 & \alpha \\ \alpha & 1 \end{pmatrix}$.

    (a) Find the eigenvalues of A.

    (b) Find the eigenspaces of A.

    (c) For which values of α is A diagonalizable?