MATH 316 Test 1 Fall '01 NAME:

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1.(10pts) Find all solutions of the linear system

$$\left| \begin{array}{llll} x & +4y & +z & =0 \\ 4x & +13y & +7z & =0 \end{array} \right|$$

2.(10pts) Specify the rank of the following matrix A by finding the reduced row-echelon form,

$$A = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 4 & 0 \\ 2 & 0 & -3 \end{bmatrix}$$

3.(10pts) Write $\begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}$ as a linear combination of $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, if possible.

4.(10pts)

a. For which choice of the constant k is the matrix $\begin{bmatrix} 2 & 3 \\ 5 & k \end{bmatrix}$ invertible?

b. For which choices of the constants a and b is the matrix $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ invertible?

5.(20pts) Let L be a line in R^2 that consists of all scalar multiples of $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$. Find the orthogonal projection as well as the reflection of $\begin{bmatrix} 4 \\ -3 \end{bmatrix}$ onto L. Find the matrix of the orthogonal projection onto L and the matrix of the reflection about L.

Solution: A unit vector in the direction of $\vec{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is $\vec{u} = \begin{bmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix}$.

$$Proj_L \begin{bmatrix} 4 \\ -3 \end{bmatrix} = \left(\begin{bmatrix} 4 \\ -3 \end{bmatrix} \cdot \begin{bmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix} \right) \begin{bmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

$$Ref_L \begin{bmatrix} 4 \\ -3 \end{bmatrix} = 2Proj_L \begin{bmatrix} 4 \\ -3 \end{bmatrix} - \begin{bmatrix} 4 \\ -3 \end{bmatrix} = \begin{bmatrix} 0 \\ 5 \end{bmatrix}.$$

To obtain the matrices of orthogonal projection and reflection, we compute

$$Proj_L \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 4/5 \\ 2/5 \end{bmatrix} \qquad Proj_L \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2/5 \\ 1/5 \end{bmatrix}.$$

Hence the matrix of orthogonal projection onto L is

$$A_{Proj} = \begin{bmatrix} 4/5 & 2/5 \\ 2/5 & 1/5 \end{bmatrix}.$$

Similarly,

$$Ref_L \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3/5 \\ 4/5 \end{bmatrix} \qquad Ref_L \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 4/5 \\ -3/5 \end{bmatrix}.$$

Hence the matrix of reflection about L is

$$A_{Ref} = \begin{bmatrix} 3/5 & 4/5 \\ 4/5 & -3/5 \end{bmatrix}.$$

6.(10pts) Interpret the following linear transformation geometrically:

$$T(\vec{x}) = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \vec{x}.$$

Find the inverse transformation of T.

Solution This is the linear transformation which rotates a vector by $\pi/4$ and followed by dilation by the factor $\sqrt{2}$. Note that $R = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$ is the rotation matrix and $D = \begin{bmatrix} \sqrt{2} & 0 \\ 0 & \sqrt{2} \end{bmatrix}$ is the dilation matrix, and $D \cdot R = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$.

Bonus(5pts) Is there a linear transformation for which $T \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $T \begin{bmatrix} -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$?

Explain your answer in full.