Solutions for Section 5.1 Homework

3. Mass is computed by $24 = m \cdot 32 \Longrightarrow m = \frac{3}{4} slug$. Spring constant is computed by $4in = \frac{1}{3} ft$ and $24 = k \cdot \frac{1}{3} \Longrightarrow k = 72 lbs/ft$. Thus the free undamped motion is the solution of
$$\frac{3}{4}x'' + 72x = 0 \qquad \text{or} \qquad x'' + 96x = 0.$$
Solving $m^2 + 96 = 0$, we get $m = \pm 4\sqrt{6}i$ and thus
$$x(t) = c_1 \cos 4\sqrt{6}t + c_2 \sin 4\sqrt{6}t.$$
Using the initial conditions $x(0) = -\frac{1}{4} ft$ and $x'(0) = 0$, we find $c_1 = -\frac{1}{4}$ and $c_2 = 0$. Hence
$$x(t) = -\frac{1}{4}\cos 4\sqrt{6}t.$$

5. Mass is computed by $20 = m \cdot 32 \Longrightarrow m = \frac{5}{8} slug$. Spring constant is computed by $6in = \frac{1}{2} ft$ and $20 = k \cdot \frac{1}{2} \Longrightarrow k = 40 lbs/ft$. Thus the free undamped motion is the solution of
$$\frac{5}{8}x'' + 40x = 0 \qquad \text{or} \qquad x'' + 64x = 0.$$
Solving $m^2 + 64 = 0$, we get $m = \pm 8i$ and thus
$$x(t) = c_1 \cos 8t + c_2 \sin 8t.$$
Using the initial conditions $x(0) = \frac{1}{2} ft$ and $x'(0) = 0$, we find $c_1 = \frac{1}{2}$ and $c_2 = 0$. Hence
$$x(t) = \frac{1}{2}\cos 4\sqrt{6}t.$$
   (a) $x(\frac{\pi}{12}) = -\frac{1}{4}, \qquad \ldots \qquad , x(\frac{9\pi}{32}) = \frac{\sqrt{2}}{4}$.
   (b) $x'(\frac{3\pi}{16}) = 4ft/sec$
   (c) Setting $x(t) = 0$ we see that $\cos 8t = 0$ which implies $t = \frac{(2k+1)\pi}{16}$.

7. Following the methods taken in the previous exercises we find that the solution for problem 6 is
$$x_1(t) = -5\sin 2t$$
and that of problem 7 is
$$x_2(t) = -10\sin t$$
   (a) $20kg$ mass in problem 7.

(b) As $x_1'(t) = -10\cos 2t$ and $x_2'(t) = -10\cos t$, $|x_1'(\frac{\pi}{4})| = 0 < 5\sqrt{2} = |x_2'(\frac{\pi}{4})|$ so the $20kg$ mass is moving faster at $\frac{\pi}{4}$ sec. Similarly at $t = \frac{\pi}{2}$, the $50kg$ mass is moving faster.

(c) Solving $-5\sin 2t = -10\sin t \Longrightarrow \sin t(\cos t - 1) = 0$, we get $t = n\pi$ for each integer n. So they are at the same position when $t = n\pi$. Also, since $x_1'(n\pi) = 10$ the $50kg$ mass is moving downward and since $x_2'(n\pi) = 10$ if n is even and $x_2'(n\pi) = -10$ if n is odd, the $20kg$ mass is moving downward at $t = n\pi$ with even n and moving upward with odd n.

9. As $m = \frac{8}{32} = \frac{1}{4}$, $\frac{1}{4}x'' + x = 0 \Longrightarrow x'' + 4x = 0$. Solving $m^2 + 4 = 0$, $m = \pm 2i$ and thus

$$x(t) = c_1 \cos 2t + c_2 \sin 2t.$$

using the initial conditions $x(0) = \frac{1}{2}$ and $x'(0) = \frac{3}{2}$, we get

$$x(t) = \frac{1}{2}\cos 2t + \frac{3}{4}\sin 2t.$$

Let $A = \sqrt{(\frac{1}{2})^2 + (\frac{3}{4})^2} = \frac{\sqrt{13}}{4}$ and we choose an angle ϕ that satisfies $\sin\phi = \frac{c_1}{A} = \frac{2}{\sqrt{13}}$ and $\cos\phi = \frac{c_2}{A} = \frac{3}{\sqrt{13}}$. Hence $\phi = 0.5880$ radian and

$$x(t) = \frac{\sqrt{13}}{4}\sin(2t + 0.5880)$$

11. $m = \frac{64}{32} = 2\,slug$ and $k = \frac{64}{0.32} = 200$. Hence

$$2x'' + 200x = 0 \Longrightarrow x'' + 100x = 0$$

and

$$x(t) = c_1 \cos 10t + c_2 \sin 10t.$$

Using $x(0) = -\frac{2}{3}$ and $x'(0) = 5$, we see that

$$x(t) = -\frac{2}{3}\cos 10t + \frac{1}{2}\sin 10t.$$

As in problem 7, let $A = \sqrt{(-\frac{2}{3})^2 + (\frac{1}{2})^2} = \frac{5}{6}$ and we choose an angle ϕ that satisfies $\sin\phi = \frac{c_1}{A} = -\frac{4}{5}$ and $\cos\phi = \frac{c_2}{A} = \frac{3}{5}$. Hence $\phi = -0.927$ radian. and

$$x(t) = \frac{5}{6}\sin(10t - 0.927)$$

(a) From the discussion above, $x(t) = \frac{5}{6}\sin(10t - 0.927)$.
(b) Amplitude is $\frac{5}{6}ft$ and period is $\frac{2\pi}{10} = \frac{\pi}{5}$.
(c) Frequency $= 1/$Amplitude so $\frac{5}{\pi}$.
(d) Solving $10t - 0.927 = 2\pi$ we get $t = 0.7207sec.$
(e) For the highest, $10t - 0.927 = \frac{\pi}{2} + 2n\pi \Longrightarrow t = \frac{1}{10}(0.927 + \frac{\pi}{2} + 2n\pi)$ and for the lowest $10t - 0.927 = -\frac{\pi}{2} + 2n\pi \Longrightarrow t = \frac{1}{10}(0.927 - \frac{\pi}{2} + 2n\pi)$.

21. Here $m = \frac{4}{32} = \frac{1}{4}\,slug$, $\beta = 1$ and $k = 2$ so that

$$\frac{1}{8}x'' + x' + 2x = 0.$$

Solving $m^2 + 8m + 16 = 0$, we get

$$x(t) = c_1e^{-4t} + c_2te^{-4t}.$$

Using $x(0) = -1$ and $x'(0) = 8$, we obtain

$$x(t) = -e^{-4t} + 4te^{-4t}.$$

Since at the equilibrium, $x(t) = 0$, solving $-e^{-4t} + 4te^{-4t} = 0 \Longrightarrow e^{-4t}(-1 + 4t) = 0$, $t = \frac{1}{4}$. Also, since at the extreme displacement, $x'(t) = 0$, we get $t = \frac{1}{2}$ and the displacement is $x(\frac{1}{2}) = e^{-\frac{1}{2}}$.

23. Here $m = 1\,kg$, $k = 10\,N/m$ and $\beta = 10$ so that

$$x'' + 10x' + 16x = 0$$

Solving $m^2 + 10m + 16 = 0$, we see that

$$x(t) = c_1e^{-2t} + c_2e^{-8t}.$$

(a) Using $x(0) = 1$ and $x'(0) = 0$, we get $x(t) = \frac{4}{3}e^{-2t} - \frac{1}{3}e^{-8t}$.
(b) Here $x(0) = 1$ and $x'(0) = -12$ so $x(t) = -\frac{2}{3}e^{-2t} - \frac{5}{3}e^{-8t}$.

25. Here $m = \frac{3.2}{32} = 0.1\,slug$, $k = 2$ and $\beta = 0.4$ and

$$0.1x'' + 0.4x' + 2x = 0.$$

Solving $m^2 + 4m + 20 = 0$, $m = -2 \pm 4i$ and

$$x(t) = e^{-2t}(c_1\cos 4t + c_2\sin 4t).$$

(a) Applying $x(0) = -1$ and $x'(0) = 0$, we get

$$x(t) = e^{-2t}(-\cos 4t - \frac{1}{2}\sin 4t).$$

(b) As before, $A = \sqrt{c_1^2 + c_2^2} = \sqrt{(-1)^2 + (-\frac{1}{2})^2} = \frac{\sqrt{5}}{2}$ and choose an angle ϕ that satisfies $\sin\phi = \frac{c_1}{A} = -\frac{2}{\sqrt{5}}$ and $\cos\phi = \frac{c_2}{A} = -\frac{1}{\sqrt{5}}$. This ϕ must be in the 3rd quadrant and thus $\phi = 4.24874$ radian and

$$x(t) = e^{-2t}\frac{\sqrt{5}}{2}\sin(4t + 4.2487).$$

(c) Examining the graph of sin function, the mass is moving upward when $4t + 4.2487 = k\pi$ with k is odd. If $k = 1$, the solution of $4t + 4.2487 = \pi$ is negative, so instead we solve $4t + 4.2487 = 3\pi$ which gives $t = 1.294 sec$.

29. $m = \frac{16}{32}$, $16 = \frac{8}{3}k \Longrightarrow k = 6$ and $\beta = \frac{1}{2}$ gives

$$\frac{1}{2}x'' + \frac{1}{2}x' + 6x = 10\cos 3t$$

Solving the auxiliary equation $\frac{1}{2}m^2 + \frac{1}{2}m + 6 = 0$, $m = -\frac{1}{2} \pm \frac{\sqrt{47}}{2}i$ and

$$x_c(t) = e^{-\frac{1}{2}t}(c_1\cos\frac{\sqrt{47}}{2}t + c_2\sin\frac{\sqrt{47}}{2}t).$$

Let $x_p(t) = A\cos 3t + B\sin 3t$ be particular solution and by the method of undetermined coefficients, we find that $A = B = \frac{10}{3}$ and

$$x(t) = e^{-\frac{1}{2}t}(c_1\cos\frac{\sqrt{47}}{2}t + c_2\sin\frac{\sqrt{47}}{2}t) + \frac{10}{3}(\cos 3t + \sin 3t).$$

Using the initial conditions, $x(0) = 2$ and $x'(0) = 0$, we find $c_1 = -\frac{4}{3}$ and $c_2 = -\frac{64}{3\sqrt{47}}$.

30. $m = 1$, $k = 5$ and $\beta = 2$. Also $x(0) = 1$ and $x'(0) = 5$. Equations of the forced damped motion is

$$x'' + 2x' + 5x = 12\cos 2t + 3\sin 2t.$$

Solving $m^2 + 2m + 5 = 0$, we get $m = -1 \pm 2i$ so that $x_c(t) = e^{-t}(c_1\cos 2t + c_2\sin 2t)$. A particular solution is sought in the form $x_p(t) = A\cos 2t + B\sin 2t$ and by the method of undetermined coefficients, we get $A = 0$ and $B = 3$, hence

$$x(t) = e^{-t}(c_1\cos 2t + c_2\sin 2t) + 3\sin 2t.$$

(a) Using the initial conditions, we find $c_1 = 1$ and $c_2 = 0$ and

$$x(t) = e^{-t}\cos 2t + 3\sin 2t.$$

(b) See handout.

31. $m = 1$, $32 = 2k \Longrightarrow k = 16$ and $\beta = 8$ so that

$$x'' + 8x' + 16x = 8\sin 4t.$$

The auxiliary equation $m^2 + 8m + 16 = 0$ has -4 as a double zero so that $x_c(t) = c_1e^{-4t} + c_2te^{-4t}$. Let $x_p(t) = A\cos 4t + B\sin 4t$ and by the method of undetermined coefficients, we find $A = -\frac{1}{4}$ and $B = 0$. Hence

$$x(t) = c_1e^{-4t} + c_2te^{-4t} - \frac{1}{4}\cos 4t.$$

Using the initial conditions $x(0) = 0$ and $x'(0) = 0$, we find $c_1 = \frac{1}{4}$ and $c_2 = 1$. Thus

$$x(t) = \frac{1}{4}e^{-4t} + te^{-4t} - \frac{1}{4}\cos 4t.$$

45. Here $L = 0.05$, $R = 2$ and $C = 0.01$ and $E(t) = 0$, so

$$Lq'' + Rq' + \frac{1}{C}q = E(t) \Longrightarrow \frac{1}{20}q'' + 2q' + 100q = 0$$

Solving this equation, we get

$$q(t) = e^{-20t}(c_1\cos 40t + c_2\sin 40t).$$

Using $q(0) = 5$ and $q'(0) = 0$, we obtain

$$q(t) = e^{-20t}(5\cos 40t + \frac{5}{2}\sin 40t).$$

The charge on the capacitor at $t = 0.01sec$ is $q(0.01) = 4.5675767$ (coulombs). Time at which the charge on the capacitor is zero can be found from solving $q(t) = 0$. This can be done by solving

$$5\cos 40t + \frac{5}{2}\sin 40t = 0$$

. This can be done as

$$5\cos 40t + \frac{5}{2}\sin 40t = 0 \Longrightarrow \tan 40t = -2$$

and $40t = -1.107$. But time must be positive so we take the solution as $40t = -1.107 + \pi$ which gives $t = 0.05086$.

46. This problem amounts to solving

$$\frac{1}{4}q'' + 20q' + 300 = 0, \qquad q(0) = 4, q'(0) = 0.$$

Solving we get

$$q(t) = e^{-40t}(4\cos 20t + 8\sin 20t).$$

49. Provided information leads us to the equation

$$q'' + 2q' + 4q = 50\cos t.$$

The steady-state charge is the particular solution of this equation. Hence, let

$$q_p(t) = A\cos t + B\sin t,$$

so that $q_p'(t) = -A\sin t + B\cos t$ and $q_p''(t) = -A\cos t - B\sin t$. Substituting them into the DE above, we obtain

$$\begin{aligned} 3A + 2B &= 50 \\ -2A + 3B &= 0 \end{aligned}$$

Solving this, we get $A = \frac{150}{13}$ and $B = \frac{100}{13}$.

$$q_p(t) = \frac{150}{13}\cos t + \frac{100}{13}\sin t.$$

As the steady-state current $i_p(t)$ is the derivative of $q_p(t)$,

$$i_p(t) = -\frac{150}{13}\sin t + \frac{100}{13}\cos t.$$

51. Here we have

$$\frac{1}{2}q'' + 20q' + 1000q = 100\sin 60t.$$

As in problem 49,

$$q_p(t) = A\cos 60t + B\sin 60t.$$

After finding $q_p'(t)$ and $q_p''(t)$ and using the method of undetermined coefficients, we get $A = -\frac{3}{52}$ and $B = -\frac{1}{26}$, so that

$$q_p(t) = -\frac{3}{52}\cos 60t - \frac{1}{26}\sin 60t.$$

The steady-state current $i_p(t)$ is

$$i_p(t) = q_p'(t) = \frac{180}{52}\sin 60t - \frac{60}{26}\cos 60t.$$

Let $c_1 = \frac{180}{52}$, $c_2 = -\frac{60}{26}$ and $A = \sqrt{c_1^2 + c_2^2} = 4.161$. Then

$$\begin{aligned} i_p(t) &= c_1 \sin 60t + c_2 \cos 60t \\ &= A(\frac{c_1}{A}\sin 60t + \frac{c_2}{A}\cos 60t) \\ &= A(\cos\phi \sin 60t + \sin\phi \cos 60t) \\ &= A\sin(60t + \phi) \end{aligned}$$

where $\cos\phi = \frac{c_1}{A} = \frac{3.462}{4.161}$ and $\sin\phi = \frac{c_2}{A} = \frac{-2.301}{4.161}$. Examining the signs of $\sin\phi$ and $\cos\phi$, ϕ must be in the 4th quadrant and $\phi = -0.5881$ radian. Thus,

$$i_p(t) = 4.161\sin(60t - 0.5881).$$