Exercises - Set 3

3.1 In applying the Euler method to an initial value problem

$$\begin{cases} y' &= f(t,y), \qquad a \le t \le b \\ y(a) &= \alpha \end{cases}$$

suppose that errors δ_i, $i = 0, 1, \ldots$ are introduced so that w_i's are computed from

$$\begin{cases} w_0 &= \alpha + \delta_0 \\ w_{i+1} &= w_i + hf(t_i, w_i) + \delta_{i+1} \qquad i = 0, 1, \ldots \end{cases}$$

If $|\delta_i| < \delta$ for all $i = 0, 1, \ldots$, then show that

$$|y(t_i) - w_i| \le \frac{1}{L}\left(\frac{hM}{2} + \frac{\delta}{h}\right)[e^{L(t_i - a)} - 1] + |\delta| e^{L(t_i - a)}$$

where, as in the class notes, L is the Lipschitz constant for f in the second variable and $|y''(t)| < M$ for all $t \in [a, b]$.

3.2 Given the IVP

$$y' = \frac{1}{t^2} - \frac{y}{t} - y^2, \qquad 1 \le t \le 2, \quad y(1) = -1$$

with exact solution $y(t) = -\frac{1}{t}$.

a. Write a computer program for Euler method (I distributed a Matlab program for this method) with $h = 0.1$ to approximate the solution.

b. How small the step size $h > 0$ must be in order for the approximation w_i to satisfy $|y(t_i) - w_i| < 10^{-6}$?

3.3 Run the program you wrote in # 3.2 for the following IVP with the different values of h.

$$y' = -100y, \qquad y(0) = 1, \quad 0 \le t \le 1$$

a. $h = 1/10$

b. $h = 1/100$

c. $h = 1/200$

d. Explain the results obtained in parts a,b and c.

3.4 Using the Taylor method of order 2, calculate the first three iterations w_1, w_2, w_3 of the problem given in # 3.2.

3.5 Repeat #3.4, using the midpoint method (Runge-Kutta of order 2).

3.6 Write a computer program that implements the Runge-Kutta of order 4 to approximate the solution of

$$y'' - 5y' - 6y = e^t - 1, \qquad y(0) = 1, y'(0) = 0.5, \quad 0 \le t \le 5.$$

3.7 Derive the implicit 3-step Adams-Moulton Method and specify its local truncation error.

$$\begin{aligned} w_0 &= \alpha \quad w_1 = \beta \quad w_2 = \gamma \\ w_{i+1} &= w_i + \frac{h}{24}[9f(t_{i+1}, w_{i+1}) + 19f(t_i, w_i) - 5f(t_{i-1}, w_{i-1}) \\ &\qquad + f(t_{i-2}, w_{i-2})] \end{aligned}$$

3.8 Solve the difference equation

$$w_{n+1} = \frac{5}{2} w_n + w_{n-1}$$

with $w_0 = w_1 = 1$. Discuss the behavior of the sequence w_n as $n \to \infty$.

3.9 In class, we examined (see handout notes) an instability of two-step method

$$w_{n+1} = w_{n-1} + 2h(-2w_n + 1)$$

applied to $y' = -2y + 1$ with $w_0 = y(0) = 1$. In the analysis, we selected w_1 by selecting the exact solution value at h, -i.e., $w_1 = \frac{1}{2}e^{-2h} + \frac{1}{2}$. Is it possible to select another value for w_1 so that $w_n \to 0$ as $n \to \infty$?

3.10 Discuss the consistency, the root condition, the stability and convergence of the following multi-step methods relative to $y' = -ky$.

a. $w_{n+1} = w_{n-1} + \frac{h}{2}(w'_{n+1} + 2w'_n + w'_{n-1})$
b. $w_{n+1} = w_n + \frac{h}{2}(w'_{n+1} + w'_n)$

3.11 In Exercise # 3.3, we study one example of stiff differential equation, -i.e., the equation that contains in its solution a transient phase as well as a steady phase. Suppose we apply the implicit difference equation

$$w_{n+1} = w_n + hw'_{n+1}$$

which is commonly known as the backward Euler equation. Apply this method to $y' = -100y$. Discuss the stability of this method. Also, modify the program you used in Exercise # 3.3 to the backward Euler method and obtain numerical results with various values of h.