Solutions to Selected Problems in Exercises - Set 1

1.4 Suppose $a_j = j$, $j = 1, 2, 3, 4$ and

$$L_{1,2}(x) = x + 1, \quad L_{2,3}(x) = 3x - 1, \quad L_{2,3,4}(2.5) = 3.$$

Find $L_{1,2,3,4}(2.5)$.

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• As $L_{123}(x) = \frac{(x-a_3)L_{12}(x)-(x-a_1)L_{23}(x)}{a_1-a_3}$, $L_{123}(2.5) = 5.75$. Then

$$L_{1234}(x) = \frac{(x-a_4)L_{123}(x) - (x-a_1)L_{234}(x)}{a_1 - a_4} = 4.375$$

1.5 A fourth-degree polynomial $L(x)$ satisfies $\triangle^4 L(0) = 24, \triangle^3 L(0) = 6$, and $\triangle^2 L(0) = 0$, where $\triangle L(x) = L(x+1) - L(x)$. Compute $\triangle^2 L(10)$.

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• Note that $\triangle^2 L(10) = L(12) - 2L(11) + L(10)$. Moreover using the Newton's forward difference formula,

$$\begin{aligned} L(10) &= f(0) + \binom{10}{1} \triangle f(0) + \binom{10}{2} \triangle^2 f(0) + \binom{10}{3} \triangle^3 f(0) + \binom{10}{4} \triangle^4 f(0) \\ &= f(0) + \binom{10}{1} \triangle f(0) + \binom{10}{2} (0) + \binom{10}{3} (6) + \binom{10}{4} (24) \end{aligned}$$

Calculate in the same way, $L(11)$ and $L(12)$. Putting these into $\triangle^2 L(10) = L(12) - 2L(11) + L(10)$ we obtain $\triangle^2(10) = 1140$.

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• We prove this by mathematical induction. For $j = 1$, we get

$$\triangle f(x) = \sum_{k=0}^{1} (-1)^{1-k} \binom{j}{k} f(x + kh) = f(x+h) - f(x)$$

and the formula is true. Assume that it is true for j, then

$$\begin{aligned}
\triangle^{j+1} f(x) &= \triangle^j \triangle f(x) \\
&= \triangle^j f(x+h) - \triangle^j f(x) \\
&= \sum_{k=0}^{j} (-1)^{j-k} \binom{j}{k} f(x+(k+1)h) - \sum_{k=0}^{j} (-1)^{j-k} \binom{j}{k} f(x+kh) \\
&= \sum_{k=0}^{j-1} (-1)^{j-k} \binom{j}{k} f(x+(k+1)h) + f(x+(j+1)h) \\
&\quad - \sum_{k=1}^{j} (-1)^{j-k} \binom{j}{k} f(x+kh) - (-1)^j f(x) \\
&= \sum_{k=1}^{j} (-1)^{j-k-1} \binom{j}{k-1} f(x+kh) + f(x+(j+1)h) \\
&\quad - \sum_{k=1}^{j} (-1)^{j-k} \binom{j}{k} f(x+kh) - (-1)^j f(x) \\
&= f(x+(j+1)h) + \sum_{k=1}^{j} (-1)^{j-k-1} \left\{ \binom{j}{k-1} + \binom{j}{k} \right\} - (-1)^j f(x) \\
&= f(x+(j+1)h) + \sum_{k=1}^{j} (-1)^{j+1-k} \binom{j+1}{k} - (-1)^j f(x) \\
&= \sum_{k=0}^{j+1} (-1)^{j+1-k} \binom{j+1}{k} f(x+kh)
\end{aligned}$$

Hence, we advanced the induction by one, which proves the formula.

1.9 Exercise # 22 P.85 of Textbook

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- (a)

$$f[a_1, \ldots, a_k] = \sum_{i=1}^{k} \frac{f(a_i)}{(a_i - a_1) \cdots (a_i - a_{i-1})(a_i - a_{i+1}) \cdots (a_i - a_k)}$$

This is proved by induction. For $k = 1$ the formula is true with $f[a_1] = f(a_1)$. Assume true for k, then

$$f[a_1,\dots,a_{k+1}] = \frac{f[a_2,\dots,a_{k+1}] - f[a_1,\dots,a_k]}{a_{k+1}-a_1}$$

$$= [\sum_{i=2}^{k+1} \frac{f(a_i)}{(a_i-a_2)\cdots(a_i-a_{i-1})(a_i-a_{i+1})\cdots(a_i-a_{k+1})}$$

$$- \sum_{i=1}^{k} \frac{f(a_i)}{(a_i-a_1)\cdots(a_i-a_{i-1})(a_i-a_{i+1})\cdots(a_i-a_k)}] \div (a_{k+1}-a_1)$$

$$= \frac{f(a_{k+1})}{(a_{k+1}-a_1)(a_{k+1}-a_2)\cdots(a_{k+1}-a_k)}$$

$$+ \frac{f(a_1)}{(a_1-a_2)(a_1-a_3)\cdots(a_1-a_{k+1})}$$

$$+ \{\sum_{i=2}^{k} \frac{f(a_i)}{(a_i-a_2)\cdots(a_i-a_{i-1})(a_i-a_{i+1})\cdots(a_i-a_{k+1})}$$

$$- \sum_{i=2}^{k} \frac{f(a_i)}{(a_i-a_1)\cdots(a_i-a_{i-1})(a_i-a_{i+1})\cdots(a_i-a_k)}\} \div (a_{k+1}-a_1)$$

Now examine each term in two summations. For example, with $i=2$, we have

$$\frac{1}{(a_{k+1}-a_1)}\left[\frac{f(a_2)}{(a_2-a_3)\cdots(a_2-a_{k+1})} - \frac{f(a_2)}{(a_2-a_1)(a_2-a_3)\cdots(a_2-a_k)}\right]$$

$$= \frac{1}{(a_{k+1}-a_1)} \frac{f(a_2)((a_2-a_1)-(a_2-a_{k+1}))}{(a_2-a_1)(a_2-a_3)\cdots(a_2-a_k)(a_2-a_{k+1})}$$

$$= \frac{f(a_2)}{(a_2-a_1)(a_2-a_3)\cdots(a_2-a_k)(a_2-a_{k+1})}$$

Hence $f[a_1,\dots,a_{k+1}] = \sum_{i=1}^{k+1} \frac{f(a_i)}{(a_i-a_1)\cdots(a_i-a_{i-1})(a_i-a_{i+1})\cdots(a_i-a_{k+1})}$ is proved.

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- (c) To show

(*) $f(x) = f[a_1] + (x-a_1)f[a_1,a_2] + \cdots + (x-a_1)\cdots(x-a_{n-1})f[a_1,\dots,a_n] + E(x)$

where

$$E(x) = p_n(x) f[a_1,\dots,a_n,x]$$

By part (c),

$$E(x) = (f[a_1,\dots,a_{n-1},x] - f[a_1,\dots,a_n])(x-a_1)\dots(x-a_{n-1})$$

Substituting this into (*), we get

$$f(x) = f[a_1] + (x - a_1)f[a_1, a_2] + \cdots + (x - a_1)\cdots(x - a_{n-2})f[a_1, \ldots, a_{n-1}] + E(x)$$

where

$$E(x) = p_{n-1}(x)f[a_1, \ldots, a_{n-1}, x]$$

Repeating this argument, we reach

$$f(x) = f[a_1] + (x - a_1)f[a_1, x]$$

which reduces to $f(x) = f[x]$ as $(x - a_1)f[a_1, x] = f[x] - f[a_1]$. Hence we showed that the equality is valid.

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- (d) From (c),

$$f(x) = f[a_1] + (x - a_1)f[a_1, a_2] + \cdots + (x - a_1)\cdots(x - a_{n-1})f[a_1, \ldots, a_n]$$

$$+p_n(x)f[a_1, \ldots, a_n, x]$$

Since $f[a_1] + (x - a_1)f[a_1, a_2] + \cdots + (x - a_1)\cdots(x - a_{n-1})f[a_1, \ldots, a_n]$ interpolates f at $a_1, a_2, \ldots, a_n$, $lim_{x\to a_k} f[a_1] + (x - a_1)f[a_1, a_2] + \cdots + (x - a_1)\cdots(x - a_{n-1})f[a_1, \ldots, a_n] = f(a_k)$, for $k = 1, 2, \ldots, n$. Also $\lim_{x\to a_k} f(x) = f(a_k)$, since f is continuous. This shows that $\lim_{x\to a_k} E(x) = 0$ for each $k = 1, 2, \ldots, n$. But

$$\begin{aligned}
\lim_{x\to a_k} E(x) &= \lim_{x\to a_k} p_n(x)f[a_1, \ldots, a_n, x] \\
&= \lim_{x\to a_k} (x - a_1)\cdots(x - a_{k-1})(x - a_{k+1})\cdots(x - a_n)(x - a_k)f[a_1, \ldots, a_n, x] \\
&= (a_k - a_1)\cdots(a_k - a_{k-1})(a_k - a_{k+1})\cdots(a_k - a_n) \\
&\qquad \times \lim_{x\to a_k} (x - a_k)f[a_1, \ldots, a_n, x]
\end{aligned}$$

Hence

$$\lim_{x\to a_k} (x - a_k)f[a_1, \ldots, a_n, x] = 0, \qquad \text{for all } k = 1, 2, \ldots, n$$

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- (e) This comes by comparing the error term we derived in class for the Lagrange polynomial

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• (h) By mathematical induction, we prove this part. Clearly true with $k = 2$. Assume that the formula is true for k. Then

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\begin{aligned}
f[a_1, \dots, a_{k+1}] &= \frac{f[a_2, \dots, a_{k+1}] - f[a_1, \dots, a_k]}{a_{k+1} - a_1} \\
&= \frac{\frac{1}{(k-1)!h^{k-1}} \triangle^{k-1} f_2 - \frac{1}{(k-1)!h^{k-1}} \triangle^{k-1} f_1}{kh} \\
&= \frac{1}{(kh)(k-1)!h^{k-1}} [\triangle^{k-1} f_2 - \triangle^{k-1} f_1] \\
&= \frac{1}{k!h^k} \triangle^k f_1
\end{aligned}
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The derivation of the Newton forward difference formula was done in class.