MATH 307 Quiz 10 NAME:

Answers start with •

1. Find the following Laplace transforms.

   (i) $L\{te^{2t}\sin t\}$

   • Since $L\{e^{2t}\sin t\} = \frac{1}{(s-2)^2+1}$, by Theorem 7.4.1,

$$L\{te^{2t}\sin t\} = -\frac{d}{ds}\frac{1}{(s-2)^2+1} = \frac{2(s-2)}{[(s-2)^2+1]^2}$$

   (ii) $L\{e^{-t} * e^t\cos t\}$

   • By Theorem 7.4.2,

$$L\{e^{-t} * e^t\cos t\} = L\{e^{-t}\}L\{e^t\cos t\} = \frac{1}{s+1}\frac{s-1}{(s-1)^2+1}$$

2. Find $L^{-1}\{\frac{1}{s(s-1)}\}$.

   • Since $\frac{1}{s} = L\{1\}$ and $\frac{1}{s-1} = L\{e^t\}$, by Theorem 7.4.2,

$$L^{-1}\{\frac{1}{s(s-1)}\} = e^t * 1 = \int_0^t e^\tau d\tau = e^t - 1$$

3. Solve the following IVP.

$$y'' + 16y = \delta(t-2\pi), \qquad y(0) = 0,\ y'(0) = 0.$$

• Taking the Laplace transforms on both sides and using the initial conditions, $y(0) = 0$ and $y'(0) = 0$, we obtain

$$s^2Y(s) + 16Y(s) = e^{-2\pi s} \Longrightarrow Y(s) = \frac{e^{-2\pi s}}{s^2+16}.$$

Since $L\{\sin 4t\} = \frac{4}{s^2+16}$, using Theorem 7.3.2 (equiv equation (15)), we get

$$y(t) = L^{-1}\{Y(s)\} = L^{-1}\{\frac{e^{-2\pi s}}{s^2+16}\} = \frac{1}{4}\sin(4(t-2\pi))U(t-2\pi).$$

4. (Bonus - 3pts) Use the Laplace transform to solve the following system of DE.

$$\frac{dx}{dt} = x - 2y$$
$$\frac{dy}{dt} = 5x - y$$
$$x(0) = -1, \quad y(0) = 2.$$

- Taking the Laplace transforms of two equations,

$$\begin{cases} sX(s)+1 &= X(s)-2Y(s) \\ sY(s)-2 &= 5X(s)-Y(s) \end{cases}$$

or

$$\begin{cases} (s-1)X(s)+2Y(s) &= -1 \\ -5X(s)+(s+1)Y(s) &= 2 \end{cases}$$

Solving for $X(s)$ and $Y(s)$, we find

$$X(s) = -\frac{s+5}{s^2+9}, \qquad Y(s) = \frac{2s-7}{s^2+9}$$

Hence

$$x(t) = L^{-1}\{X(s)\} = -L^{-1}\{\frac{s}{s^2+9}\} - \frac{5}{3}L^{-1}\{\frac{3}{s^2+9}\} = -\cos 3t - \frac{5}{3}\sin 3t$$

and

$$y(t) = L^{-1}\{Y(s)\} = L^{-1}\{\frac{2s}{s^2+9}\} - \frac{7}{3}L^{-1}\{\frac{3}{s^2+9}\} = 2\cos 3t - \frac{7}{3}\sin 3t.$$