MATH 307 Test 2 Solutions:

1.(10pts) Solve the following DE by variation of parameters.

$$y'' - 4y = \frac{e^{2x}}{x}$$

- Solving $m^2 - 4 = 0$, $y_c(x) = c_1 e^{2x} + c_2 e^{-2x}$. We seek a particular solution y_p by variation of parameters. Note that

$$W = \begin{vmatrix} e^{2x} & e^{-2x} \\ 2e^{2x} & -2e^{-2x} \end{vmatrix} = -4, \quad W_1 = \begin{vmatrix} 0 & e^{-2x} \\ \frac{e^{2x}}{x} & -2e^{-2x} \end{vmatrix} = -\frac{1}{x}, \quad W_2 = \begin{vmatrix} e^{2x} & 0 \\ 2e^{2x} & \frac{e^{2x}}{x} \end{vmatrix} = \frac{e^{4x}}{x}$$

Hence

$$u_1' = \frac{W_1}{W} = \frac{1}{4x} \Longrightarrow u_1 = \frac{1}{4}\ln x$$

and

$$u_2' = \frac{W_2}{W} = \frac{-e^{4x}}{4x} \Longrightarrow u_2 = -\frac{1}{4}\int_{x_0}^{x} \frac{e^{4t}}{t}dt$$

$y_p = u_1 y_1 + u_2 y_2$ implies

$$y = y_c + y_p = c_1 e^{2x} + c_2 e^{-2x} + \frac{1}{4}e^{2x}\ln x - \frac{1}{4}e^{-2x}\int_{x_0}^{x} \frac{e^{4t}}{t}dt.$$

2.(10pts) Solve the following Cauchy-Euler equation.

$$3x^2y'' + 6xy' + y = 0$$

- Substituting $y = x^m$, $y' = mx^{m-1}$ and $y'' = m(m-1)x^{m-2}$ into DE, we obtain

$$[3m(m-1) + 6m + 1]x^m = 0 \Longrightarrow 3m(m-1) + 6m + 1 = 0 \Longrightarrow 3m^2 + 3m + 1 = 0$$

Solving

$$m = -\frac{1}{2} \pm \frac{\sqrt{3}}{6}i.$$

Using equation (4), page 165,

$$y(x) = x^{-\frac{1}{2}}[c_1 \cos(\frac{\sqrt{3}}{6}\ln x) + c_2 \sin(\frac{\sqrt{3}}{6}\ln x)].$$

3.(10pts) Solve the following Cauchy-Euler equation by first transforming the equation to a differential equation with constant coefficients using the transformation $x = e^t$.

$$x^2y'' - 9xy' + 25y = 0.$$

- With the transformation $x = e^t$, we get

$$\frac{dy}{dx} = \frac{1}{x}\frac{dy}{dt}, \quad \frac{d^2y}{dx^2} = \frac{1}{x^2}(\frac{d^2y}{dt^2} - \frac{dy}{dt}). \qquad \text{see example 6, page 167}$$

The DE above transforms to

$$\frac{d^2y}{dt^2} - \frac{dy}{dt} - 9\frac{dy}{dt} + 25y = 0 \Longrightarrow \frac{d^2y}{dt^2} - 10\frac{dy}{dt} + 25 = 0.$$

The solution to the last equation is

$$y(t) = c_1e^{5t} + c_2te^{5t}$$

and using $x = e^t$, we see that

$$y(x) = c_1x^5 + c_2x^5 \ln x.$$

4.(10pts) A mass weighing 4 pounds is attached to a spring whose spring constant is 16 lbs/ft. What is the period of simple harmonic motion?

- Here $k = 16$ and $m = 4/32 = \frac{1}{8}$ slug. Hence the DE describing the motion of this free undamped motion is

$$\frac{1}{8}x'' + 16x = 0.$$

Solving the auxiliary equation $m^2 + 128 = 0$, we get

$$x(t) = c_1 \cos 8\sqrt{2}x + c_2 \sin 8\sqrt{2}x.$$

The period is $\frac{2\pi}{8\sqrt{2}} = \frac{\sqrt{2}}{8}\pi$ second.

5.(10pts) A 4-foot spring measures 8 feet long after a mass weighing 8 pounds is attached to it. The medium through which the mass moves offers a damping force numerically equal to $\sqrt{2}$ times the instantaneous velocity. Find the equation of motion if the mass is initially released from the equilibrium position with a downward velocity of 5 ft/s.

• From $8 = 4k$, we see that $k = 2$ is the spring constant. Also $m = \frac{8}{32} = \frac{1}{4}$ slug. The DE of motion is

$$\frac{1}{4}x'' + \sqrt{2}x' + 2x = 0$$

The auxiliary equation is $\frac{1}{4}m^2 + \sqrt{2}m + 2 = 0$ and its solution is

$$m = -2\sqrt{2}, \qquad \text{a double root.}$$

Hence

$$x(t) = c_1 e^{-2\sqrt{2}t} + c_2 t e^{-2\sqrt{2}t}.$$

Using $x(0) = 0$ we find $c_1 = 0$. As $x'(t) = c_2(e^{-2\sqrt{2}t} - 2\sqrt{2}te^{-2\sqrt{2}t})$, $x'(0) = 5$ implies that $c_2 = 5$. Thus,

$$x(t) = 5te^{-2\sqrt{2}t}.$$

6.(20pts) Find the Laplace transform $L\{f(t)\}$ of the following function $f(t)$.

(i) $f(t) = t^2 - e^{-3t} + 4$

- $L\{t^2 - e^{-3t} + 4\} = \frac{2}{s^3} - \frac{1}{s+3} + \frac{4}{s}$

(ii) $f(t) = \cos 6t + \sinh 2t$

- $L\{\cos 6t + \sinh 2t\} = \frac{s}{s^2+36} + \frac{2}{s^2-4}$

(iii) $f(t) = e^{2-t}U(t-2)$

- $L\{e^{2-t}U(t-2)\} = L\{e^{-(t-2)}U(t-2)\} = e^{-2s}\frac{1}{s+1}$, with last equality from Theorem 7.3.2 with $f(t) = e^{-t}$.

(iv) $f(t) = \int_0^{\tau} \tau \sin \tau d\tau$

- Typo: $f(t) = \int_0^t \tau \sin \tau d\tau$ which is $t \sin t * 1$, so by Theorem 7.4.2,

$$L\{\int_0^t \tau \sin \tau d\tau\} = L\{t \sin t\}L\{1\} = (-\frac{d}{ds}\frac{1}{s^2+1})\frac{1}{s} = \frac{2s}{(s^2+1)^2}\frac{1}{s}.$$

(v) $f(t) = te^t \sin 2t$

- Since $L\{e^t \sin 2t\} = \frac{2}{(s-1)^2+4}$, (Theorem 7.3.1),

$$L\{te^t \sin 2t\} = -\frac{d}{ds}\frac{2}{(s-1)^2+4} = \frac{4(s-1)}{[(s-1)^2+4]^2},$$

where the 1st equality was by Theorem 7.4.1.

7.(20pts) Find the inverse Laplace transform $L^{-1}\{F(s)\}$ of the following function $F(s)$.

(i) $F(s) = \frac{1}{s^4} + \frac{4}{s-2}$

- $L^{-1}\{F(s)\} = \frac{1}{6}L^{-1}\{\frac{3!}{s^4}\} + 4L^{-1}\{\frac{1}{s-2}\} = \frac{1}{6}t^3 + 4e^{2t}$

(ii) $F(s) = \frac{s+2}{s^2+4}$

- $L^{-1}\{F(s)\} = L^{-1}\{\frac{s}{s^2+4}\} + L^{-1}\{\frac{2}{s^2+4}\} = \cos 2t + \sin 2t.$

(iii) $F(s) = \frac{1}{s^2+2s+4}$

- $L^{-1}\{\frac{1}{s^2+2s+4}\} = L^{-1}\{\frac{1}{(s+1)^2+3}\} = \frac{1}{\sqrt{3}}L^{-1}\{\frac{\sqrt{3}}{(s+1)^2+3}\} = \frac{1}{\sqrt{3}}e^{-t}\sin\sqrt{3}t$, with the last equality from Theorem 7.3.1.

(iv) $F(s) = \frac{e^{-2s}}{s^2(s-1)}$

- Note that by partial fractions we obtain $\frac{1}{s^2(s-1)} = \frac{1}{s} - \frac{1}{s^2} - \frac{1}{s-1}$, hence

$$L^{-1}\{\frac{e^{-2s}}{s^2(s-1)}\} = L^{-1}\{e^{-2s}(\frac{1}{s} - \frac{1}{s^2} - \frac{1}{s-1})\} = (1 - (t-2) - e^{t-2})U(t-2).$$

(v) $F(s) = e^{-7s}$

- $L^{-1}\{e^{-7s}\} = \delta(t-7)$ by Theorem 7.5.1.

8.(10pts) Solve by the Laplace transform

$$y'' + 4y = f(t), \qquad y(0) = 0,\ y'(0) = -1,$$

where

$$f(t) = \begin{cases} 1, & 0 \le t < 1 \\ 0, & t \ge 1 \end{cases}$$

• Since $L\{y''\} = s^2Y(s) - sy(0) - y'(0)$, $L\{y\} = Y(s)$, $f(t) = 1 - U(t-1)$ and $L\{1 - U(t-1)\} = \frac{1}{s} - \frac{e^{-s}}{s}$, the Laplace transform of DE is

$$(s^2+4)Y(s) = \frac{1}{s} - \frac{e^{-s}}{s} - 1$$

or

$$Y(s) = \frac{1-s}{s(s^2+4)} - \frac{e^{-s}}{s(s^2+4)}$$

Partial fraction yields

$$\frac{1-s}{s(s^2+4)} = \frac{1}{4s} - \frac{\frac{1}{4}s+1}{s^2+4},$$

so that

$$Y(s) = \frac{1}{4s} - \frac{\frac{1}{4}s+1}{s^2+4} - e^{-s}\left(\frac{1}{4s} - \frac{\frac{1}{4}s+1}{s^2+4}\right).$$

Hence,

$$y(t) = L^{-1}\{Y(s)\} = \frac{1}{4} - \frac{1}{4}\cos 2t - \frac{1}{2}\sin 2t -$$

$$-U(t-1)\left[\frac{1}{4} - \frac{1}{4}\cos 2(t-1) - \frac{1}{2}\sin 2(t-1)\right]$$