Sample Final Exam - Solutions
Final Exam: Monday, Dec. 8, 2008, 3:35-6:45pm

1. Solve the following DE.

(i) $\csc y\,dx + \sec^2 x\,dy = 0$ (separable)

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$$\csc y\,dx + \sec^2 x\,dy = 0 \implies \frac{dy}{\csc y} = -\frac{1}{\sec^2 x}dx$$
$$\implies \sin y\,dy = -\cos^2 x\,dx \implies -\cos y = -\int \frac{1+\cos 2x}{2}\,dx$$
$$\implies \cos y = \frac{1}{2}x + \frac{1}{4}\sin 2x + C$$

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(ii) $(x+1)\frac{dy}{dx} + (x+2)y = 2xe^{-x}$ (linear)

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$$(x+1)\frac{dy}{dx} + (x+2)y = 2xe^{-x} \implies y' + \frac{x+2}{x+1}y = \frac{2x}{x+1}e^{-x}$$

The integrating factor is

$$e^{\int \frac{x+2}{x+1}dx} = e^{\int 1+\frac{1}{x+1}dx} = e^{x+\ln|x+1|} = e^x(x+1)$$

Multiplying the DE $y' + \frac{x+2}{x+1}y = \frac{2x}{x+1}e^{-x}$ by I.F.

$$[e^x(x+1)y]' = 2x \implies e^x(x+1)y = x^2 + C,\ x \in (-1,\infty)$$

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(iii) $(e^{2y} - y\cos xy)\,dx + (2xe^{2y} - x\cos xy + 2y)\,dy = 0$ (exact)

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$$\frac{\partial M}{\partial y} = 2e^{2y} + xy\sin xy - \cos xy = \frac{\partial N}{\partial x}$$

so the DE is exact. Hence,

$$\frac{\partial f}{\partial y} = 2xe^{2y} - x\cos xy + 2y \implies f(x,y) = xe^{2y} - \sin xy + y^2 + h(x)$$

Taking the partial derivative of this f with respect to x, we get

$$\frac{\partial f}{\partial x} = e^{2y} - y\cos xy + h'(x) = M(x,y) \implies h'(s) = 0 \implies h(x) = C$$

Hence,

$$2xe^{2y} - x\cos xy + 2y \Longrightarrow f(x,y) = xe^{2y} - \sin xy + y^2 + C = 0$$

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(iv) $-y\,dx + (x + \sqrt{xy})\,dy = 0$ (homogeneous)

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$$-y\,dx + (x + \sqrt{xy})\,dy = 0 \Longrightarrow x(-\frac{y}{x}\,dx + (1 + \sqrt{\frac{y}{x}})\,dy) = 0$$

Let $u = \frac{y}{x}$ so $dy = xdu + udx$. Then

$$-\frac{y}{x}\,dx + (1 + \sqrt{\frac{y}{x}})\,dy = 0 \Longrightarrow -udx + (1 + \sqrt{u})(xdu + udx) = 0$$

$$\Longrightarrow \frac{dx}{x} + \frac{1+\sqrt{u}}{u\sqrt{u}}du = 0 \Longrightarrow \ln|x| - 2u^{-1/2} + \ln u + C = 0$$

$$\ln|x| - 2(\frac{y}{x})^{-1/2} + \ln\frac{y}{x} + C = 0 \Longrightarrow \ln|x| - 2(\frac{y}{x})^{-1/2} + \ln|y| - \ln|x| + C = 0$$

$$\Longrightarrow \ln|y| + C = 2\sqrt{\frac{x}{y}} \Longrightarrow 4x = y(\ln y + C)^2$$

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(v) $\frac{dy}{dx} = y(xy^3 - 1)$ (Bernoulli)

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$$\frac{dy}{dx} = y(xy^3 - 1) \Longrightarrow \frac{dy}{dx} = xy^4 - y \Longrightarrow \frac{dy}{dx} + y = xy^4$$

Let $u = y^{1-4} = y^{-3}$ so that $y = u^{-1/3}$ and $\frac{dy}{dx} = -\frac{1}{3}u^{-4/3}\frac{du}{dx}$.

$$\frac{dy}{dx} + y = xy^4 \Longrightarrow -\frac{1}{3}\frac{1}{u^{4/3}}\frac{du}{dx} + u^{-1/3} = xu^{-4/3} \Longrightarrow \frac{du}{dx} - 3u = -3x$$

The last DE is linear and its integrating factor is $e^{-\int 3dx} = e^{-3x}$. Then

$$[e^{-3x}u]' = -3xe^{-3x} \Longrightarrow e^{-3x}u = -3(-\frac{1}{3}xe^{-3x} - \frac{1}{9}e^{-3x} + C)$$

$$\Longrightarrow u = -3(-\frac{1}{3}x - \frac{1}{9} + Ce^{3x}) \Longrightarrow y^{-3} = x + \frac{1}{3} + Ce^{3x}$$

2. A small metal bar, whose initial temperature was 20°C, is dropped into a large container of boiling water. How long will it take the bar to reach 90°C if it is known that its temperature increases 2° in 1 second?
(You should review all HW problems in Section 3.1)

- From the Newton's law of cooling,

$$\frac{dT}{dt} = k(T-100) \Longrightarrow \frac{dT}{T-100} = k\,dt$$

$$\Longrightarrow \ln(T-100) = kt + C \Longrightarrow T = Ce^{kt} + 100$$

Since $T(0) = 20\text{C}^\circ$, $T(0) = C + 100 = 20 \Longrightarrow C = -80$. Hence $T(t) = -80e^{kt} + 100$. The heat conductivity constant k can be found from the fact that temperature increases 2° in 1 second. Namely, solve for k the equation

$$22 = -80e^k + 100.$$

$22 = -80e^k + 100 \Longrightarrow -78 = -80e^k \Longrightarrow k = \ln\frac{78}{80} \approx -0.0253178$. Time it takes for the bar to reach 90° can be found by solving $90 = -80e^{-0.0253178\,t} + 100$. $t = 82.1$ seconds.

3. Find the general solution of the following DE.

(i) $y''' - 5y'' + 3y' + 9y = 0$

• $m^3 - 5m^2 + 3m + 9 = 0 \Longrightarrow (m+1)(m^2 - 6m + 9) = 0 \Longrightarrow (m-1)(m-3)^2 = 0$
Hence roots are $m = -1, 3, 3$.

$$y(x) = c_1 e^{-x} + c_2 e^{3x} + c_3 x e^{3x}.$$

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(ii) $y'' - y' + y = 2\sin 3x$

• $m^2 - m + 1 = 0 \Longrightarrow m = \frac{1}{2} \pm \frac{\sqrt{3}}{2} i$ so that

$$y_c = c_1 e^{\frac{1}{2}x} \cos \frac{\sqrt{3}}{2} x + c_2 e^{\frac{1}{2}x} \sin \frac{\sqrt{3}}{2} x$$

We try a particular solution in the form $y_p = A\cos 3x + B\sin 3x$. Differentiating y_p twice, we get $y_p' = -3A\sin 3x + 3B\cos 3x$ and $y_p'' = -9A\cos 3x - 9B\sin 3x$. Thus,

$$y_p'' - y_p' + y_p = (-8A - 3B)\cos 3x + (3A - 8B)\sin 3x$$

Setting the last expression equal $2\sin 3x$ and comparing the coefficients,

$$-8A - 3B = 0, \qquad 3A - 8B = 2$$

Solving $A = \frac{6}{73}$ and $B = -\frac{16}{73}$, so

$$y_p = \frac{6}{73}\cos 3x - \frac{16}{73}\sin 3x$$

The general solution is

$$y = y_c + y_p = c_1 e^{\frac{1}{2}x} \cos \frac{\sqrt{3}}{2} x + c_2 e^{\frac{1}{2}x} \sin \frac{\sqrt{3}}{2} x + \frac{6}{73}\cos 3x - \frac{16}{73}\sin 3x$$

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(iii) $y'' + 3y = -48x^2e^{3x}$

• $m^2 + 3 = 0 \Longrightarrow m = \pm\sqrt{3}i$ so

$$y_c = c_1 \cos\sqrt{3}x + c_2 \sin\sqrt{3}x$$

We try a particular solution in the form $y_p = (Ax^2 + Bx + C)e^{3x}$. Taking the derivatives, $y_p' = (3Ax^2+(2A+3B)x+(B+3C))e^{3x}$ and $y_p'' = (9Ax^2+(12A+9B)x+(2A+6B+9C))e^{3x}$. Hence,

$$y_p'' + 3y_p = [12Ax^2 + (12A + 12B)x + (2A + 6B + 12C)]e^{3x} = -48x^2e^{3x}$$

This implies that $12A = -48$, $12A+12B = 0$ and $2A+6B+12C = 0$, which gives $A = -4$, $B = 4$ and $C = -\frac{4}{3}$ and

$$y_p = (-4x^2 + 4B - \frac{4}{3})3^{3x}$$

The general solution is

$$y = y_c + y_p = c_1 \cos\sqrt{3}x + c_2 \sin\sqrt{3}x + (-4x^2 + 4B - \frac{4}{3})3^{3x}$$

4. Solve

(i) $y'' + y = \sin x$ (variation of parameters)

• $m^2 + 1 = 0 \Longrightarrow m = \pm i$ so the fundamental solutions are $y_1 = \cos x$ and $y_2 = \sin x$. using the variation of parameters formula in Section 4.6,

$$u_1' = \frac{\begin{vmatrix} 0 & \sin x \\ \sin x & \cos x \end{vmatrix}}{\begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix}} = \frac{-\sin^2 x}{1} = -\sin^2 x,$$

$$u_2' = \frac{\begin{vmatrix} \cos x & 0 \\ -\sin x & \sin x \end{vmatrix}}{\begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix}} = \frac{\sin x \cos x}{1} = \sin x \cos x$$

Integrating u_1' and u_2',

$$u_1 = -\int \sin^2 x dx = -\int \frac{1-\cos 2x}{2} dx = -\frac{1}{2}x + \frac{1}{4}\sin 2x$$

$$u_2 = \int \sin x \cos x dx = \frac{1}{2}\sin^2 x$$

Finally,

$$\begin{aligned} y = y_c + y_p &= c_1 \cos x + c_2 \sin x + [\frac{1}{4}\sin 2x - \frac{1}{2}x]\cos x + \frac{1}{2}\sin^2 x \sin x \\ &= c_1 \cos x + c_2 \sin x + \frac{1}{2}\sin x \cos^2 x - \frac{1}{2}x\cos x + \frac{1}{2}(1-\cos^2 x)\sin x \\ &= c_1 \cos x + c_2 \sin x - \frac{1}{2}x\cos x \end{aligned}$$

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(ii) $x^2y'' - 3xy' + 3y = 2x^4e^x$ (nonhomogeneous Cauchy-Euler equation)

• First solve the homogeneous equation $x^2y'' - 3xy' + 3y = 0$ by substituting $y = x^m$ and finding the roots of auxiliary equation $m(m-1) - 3m + 3 = 0 \implies m^2 - 4m + 3 = 0 \implies (m-1)(m-3) = 0$, we see that $y_1 = x$ and $y_2 = x^3$. Transforming the DE into the standard form

$$y'' - \frac{3}{x}y' + \frac{3}{x^2}y = 2x^2e^x$$

As in the previous problem,

$$u_1' = \frac{\begin{vmatrix} 0 & x^3 \\ 2x^2e^x & 3x^2 \end{vmatrix}}{\begin{vmatrix} x & x^3 \\ 1 & 3x^2 \end{vmatrix}} = \frac{-2x^5e^x}{2x^3} = -x^2e^x,$$

$$u_2' = \frac{\begin{vmatrix} x & 0 \\ 1 & 2x^2e^x \end{vmatrix}}{\begin{vmatrix} x & x^3 \\ 1 & 3x^2 \end{vmatrix}} = \frac{2x^3e^x}{2x^3} = e^x.$$

For u_1, integrate by parts $u_1' = -x^2e^x$ twice to get

$$u_1 = -(x^2 - 2x + 2)e^x$$

Clearly $u_2 = e^x$. Finally, the general solution is

$$\begin{aligned} y &= c_1x + c_2x^3 + u_1y_1 + u_2y_2 \\ &= c_1x + c_2x^3 - (x^2 - 2x + 2)xe^x + x^3e^x \end{aligned}$$

5. A mass of 1-slug, when attached to a spring, stretches it 2 feet and then comes to rest in the equilibrium position. Starting at $t = 0$, an external force equal to $f(t) = e^{-t} \sin 4t$ is applied to the system. Find the equation of motion if the surrounding medium offers a damping force that is numerically equal to 8 times the instantaneous velocity. (You should review all HW problems in Section 5.1)

• $m = 1$slug. Compute the spring constant from $mg = kx \Longrightarrow 32 = k2 \Longrightarrow k =$ 16lbs/ft. Also $\beta = 8$. Hence the DE describing the motion is

$$x'' + 8x' + 16x = e^{-t} \sin 4t.$$

The solution to the corresponding homogeneous equation is found by solving $m^2+8m+16 = 0$, which gives $(m+4)^2 = 0$ or $m = -4$. Hence,

$$x_c(t) = c_1 e^{-4t} + c_2 t e^{-4t}.$$

Since the forcing function is $e^{-t} \sin 4t$, we try

$$x_p(t) = Ae^{-t} \cos 4t + Be^{-t} \sin 4t = e^{-t}[A \cos 4t + B \sin 4t]$$

Differentiating twice, we see that

$$x_p'(t) = e^{-t}[(4A - B) \cos 4t - (A + 4B) \sin 4t]$$

and

$$x_p''(t) = e^{-t}[(-8A - 15B) \cos 4t + (-15A + 8B) \sin 4t]$$

Substituting these derivatives into the DE,

$$x_p'' + 8x_p' + 16x_p = e^{-t}[(24A - 7B) \cos 4t + (-7A - 24B) \sin 4t]$$

This implies that

$$\begin{aligned} 24A - 7B &= 0 \\ -7A - 24B &= 1 \end{aligned}$$

whose solutions are $A = -\frac{7}{625}$ and $B = -\frac{24}{625}$. Thus, the general solution is give by

$$x(t) = x_c(t) + x_p(t) = c_1 e^{-4t} + c_2 t e^{-4t} + e^{-t}[-\frac{7}{625} \cos 4t - \frac{24}{625} \sin 4t]$$

6. Find the Laplace transform of the following functions.

(i) $f(t) = \begin{cases} 4 & 0 \le t < 2 \\ 0, & t \ge 2 \end{cases}$

- Since $f(t) = 4 - 4U(t-2)$, $L\{f(t)\} = \frac{4}{s} - 4\frac{e^{-2s}}{s}$.

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(ii) $f(t) = e^{-t}\sinh t$

- Since $L\{\sinh t\} = \frac{1}{s^2-1}$, $L\{e^{-t}\sinh t\} = \frac{1}{(s+1)^2-1}$

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(iii) $f(t) = \int_0^t \tau e^{t-\tau}\,d\tau$

- $L\{\int_0^t \tau e^{t-\tau}\,d\tau\} = L\{t * e^t\} = L\{t\}L\{e^t\} = \frac{1}{s^2}\frac{1}{s-1}$

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(iv) $f(t) = t(e^t + e^{-2t})^2$

- $L\{t(e^t + e^{-2t})^2\} = L\{(e^{2t} + 2e^{-t} + e^{-4t})t\} = \frac{1}{(s-2)^2} + \frac{2}{(s+1)^2} + \frac{1}{(s+4)^2}$

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(v) $f(t) = \cos 2tU(t-\pi)$

Using equation (16), P. 276,

$$L\{\cos 2tU(t-\pi)\} = e^{-\pi s}L\{\cos 2(t+\pi)\} = e^{-\pi s}L\{\cos 2t\} = e^{-\pi s}\frac{s}{s^2+4}$$

7. Find the inverse Laplace transform of the following functions.

(i) $F(s) = \frac{s}{s^2+2s-3}$

• By partial fraction $\frac{s}{s^2+2s-3} = \frac{3/4}{s+3} + \frac{1/4}{s-1}$, hence

$$L^{-1}\{\frac{s}{s^2+2s-3}\} = L^{-1}\{\frac{3/4}{s+3}\} + L^{-1}\{\frac{1/4}{s-1}\} = \frac{3}{4}e^{-3t} + \frac{1}{4}e^{t}$$

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(ii) $F(s) = \frac{s}{s^2+4s+5}$

• Note that

$$\frac{s}{s^2+4s+5} = \frac{s}{(s+2)^2+1} = \frac{s+2}{(s+2)^2+1} - \frac{2}{(s+2)^2+1}$$

Hence,

$$L^{-1}\{\frac{s}{s^2+4s+5}\} = L^{-1}\{\frac{s+2}{(s+2)^2+1}\} - L^{-1}\{\frac{2}{(s+2)^2+1}\} = e^{-2t}\cos t - 2e^{-2t}\sin t$$

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(iii) $F(s) = \frac{(1+e^{-2s})^2}{s+2}$

• Since $\frac{(1+e^{-2s})^2}{s+2} = \frac{1+2e^{-2s}+e^{-4s}}{s+2} = \frac{1}{s+2} + \frac{2e^{-2s}}{s+2} + \frac{e^{-4s}}{s+2}$,

$$L^{-1}\{\frac{(1+e^{-2s})^2}{s+2}\} = e^{-2t} + 2e^{-2(t-2)}U(t-2) + e^{-2(t-4)}U(t-4)$$

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(iv) $F(s) = \frac{1}{(s^2+1)^2}$

• Note that $L\{\sin t\} = \frac{1}{s^2+1}$, hence

$$L^{-1}\{\frac{1}{(s^2+1)^2}\} = \sin t * \sin t = \int_0^t \sin\tau \sin(t-\tau)d\tau$$

$$= \int_0^t \frac{1}{2}[\cos(2\tau - t) - \cos t]d\tau = \frac{1}{2}[\frac{1}{2}\sin(2\tau - t) - \tau\cos t]|_0^t$$

$$= \frac{1}{2}[\frac{1}{2}(\sin t - \sin(-t)) - t\cos t] = \frac{1}{2}[\sin t - t\cos t]$$

In the third equality above, we used $\sin A \sin B = \frac{1}{2}[\cos(A-B) - \cos(A+B)]$.

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(v) $F(s) = \frac{s}{(s^2+1)^2}$

• Note that $\frac{s}{(s^2+1)^2} = -\frac{d}{ds}\frac{1}{2(s^2+1)} = L\{\frac{1}{2}t\sin t\}$ hence

$$L^{-1}\{\frac{s}{(s^2+1)^2}\} = \frac{1}{2}t\sin t$$

8. Use the Laplace transform to solve the following initial-value problem.

$$y'' + 2y' = \delta(t-1), \qquad y(0) = 0,\, y'(0) = 1$$

- $s^2Y(s) - sy(0) - y'(0) + 2[sY(s) - y(0)] = e^{-s} \Longrightarrow (s^2 + 2s)Y(s) = e^{-s} + 1$

$$Y(s) = \frac{e^{-s}}{s^2 + 2s} + \frac{1}{s^2 + 2s}$$

By partial fractions,

$$\frac{1}{s^2 + 2s} = \frac{1/2}{s} - \frac{1/2}{s+2}$$

and hence

$$y(t) = L^{-1}\{Y(s)\} = L^{-1}\{\frac{1/2}{s} - \frac{1/2}{s+2} + e^{-s}(\frac{1/2}{s} - \frac{1/2}{s+2})\}$$

$$= \frac{1}{2} - \frac{1}{2}e^{-2t} + \frac{1}{2}U(t-1) - \frac{1}{2}e^{-2(t-1)}U(t-1).$$

9. Find two power series solutions of the following DE about the ordinary point $x = 0$,

$$y'' + x^2y' + xy = 0.$$

• We seek a solution in the form $y(x) = \sum_{n=0}^{\infty} c_n x^n$. Since $y'(x) = \sum_{n=1}^{\infty} c_n n x^{n-1}$ and $y''(x) = \sum_{n=2}^{\infty} c_n n(n-1)x^{n-2}$,

$$xy = \sum_{n=0}^{\infty} c_n x^{n+1}, \quad x^2y' = \sum_{n=1}^{\infty} c_n n x^{n+1}$$

$$y'' + x^2y' + xy = 0 \Longrightarrow \sum_{n=2}^{\infty} c_n n(n-1)x^{n-2} + \sum_{n=1}^{\infty} c_n n x^{n+1} + \sum_{n=0}^{\infty} c_n x^{n+1} = 0$$

Note that the first series starts with the term x^0, the second with x^2 and the third with x^1. Hence all terms starting with x^2 onward are commonly shared in all three series, and therefore they can be combined into a single series. Separate the the first two terms from the first series and the the first term from the third series, we get

$$2c_2 + (c_0 + 6c_3)x + \sum_{n=4}^{\infty} c_n n(n-1)x^{n-2} + \sum_{n=1}^{\infty} c_n n x^{n+1} + \sum_{n=1}^{\infty} c_n x^{n+1} = 0$$

Let $k = n-2$, $k = n+1$ and $k = n+1$ respectively in the 1st, 2nd and 3rd series, we get

$$2c_2 + (c_0 + 6c_3)x + \sum_{k=2}^{\infty} c_{k+2}(k+2)(k+1)x^k + \sum_{k=2}^{\infty} c_{k-1}(k-1)x^k + \sum_{k=2}^{\infty} c_{k-1}x^k = 0$$

$$2c_2 + (c_0 + 6c_3)x + \sum_{k=2}^{\infty} [c_{k+2}(k+2)(k+1) + c_{k-1}(k-1) + c_{k-1}]x^k = 0$$

or

$$2c_2 + (c_0 + 6c_3)x + \sum_{k=2}^{\infty} [c_{k+2}(k+2)(k+1) + c_{k-1}k]x^k = 0$$

By setting all the coefficients equal 0, we get the following recurrence relations;

$$\begin{aligned} c_2 &= 0 \\ c_3 &= -\frac{1}{6}c_0 = -\frac{1}{3!}c_0 \\ c_{k+2} &= -\frac{kc_{k-1}}{(k+2)(k+1)}, \quad k \geq 2 \end{aligned}$$

Using the recurrence relation, we see that

$$\begin{aligned}
k = 2 \qquad & c_4 = -\frac{2c_1}{4 \cdot 3} = -\frac{2^2}{4!}c_1 \\
k = 3 \qquad & c_5 = -\frac{3c_2}{5 \cdot 4} = 0 \\
k = 4 \qquad & c_6 = -\frac{4c_3}{6 \cdot 5} = (-\frac{4}{6 \cdot 5})(-\frac{1}{3!}c_0) = \frac{4^2}{6!}c_0 \\
k = 5 \qquad & c_7 = -\frac{5c_4}{7 \cdot 6} = (-\frac{5}{7 \cdot 6})(\frac{-2^2}{4!})c_1 = \frac{5^2 2^2}{7!}c_1 \\
k = 6 \qquad & c_8 = -\frac{6}{8 \cdot 7}c_5 = 0 \\
k = 7 \qquad & c_9 = -\frac{7}{9 \cdot 8}c_6 = -\frac{7}{9 \cdot 8}\frac{4^2}{6!}c_0 = -\frac{7^2 4^2}{9!}c_0 \\
\vdots \qquad & \vdots
\end{aligned}$$

Thus,

$$\begin{aligned}
y(x) &= c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + c_5 x^5 + c_6 x^6 + \cdots \\
&= c_0 + c_1 x - \frac{1}{3!}c_0 x^3 - \frac{2^2}{4!}c_1 x^4 + \frac{4^2}{6!}c_0 x^6 + \frac{5^2 2^2}{7!}c_1 x^7 - \frac{7^2 4^2}{9!}c_0 x^9 - \cdots \\
&= c_0[1 - \frac{1}{3!}x^3 + \frac{4^2}{6!}x^6 - \frac{7^2 4^2}{9!} + \cdots] + c_1[x - \frac{2^2}{4!}x^4 + \frac{5^2 2^2}{7!}x^7 - \cdots]
\end{aligned}$$