Math 316 - Test 1- Fall '09 - Solution

Show all your work.

Each problem is worth 10 points.

1. Find the reduced row echelon form, $rref(A)$, of

$$A = \begin{bmatrix} 0 & 0 & 1 & -1 & -1 \\ 2 & 4 & 2 & 4 & 2 \\ 2 & 4 & 3 & 3 & 3 \\ 3 & 6 & 6 & 3 & 6 \end{bmatrix}$$

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$$\begin{bmatrix} 0 & 0 & 1 & -1 & -1 \\ 2 & 4 & 2 & 4 & 2 \\ 2 & 4 & 3 & 3 & 3 \\ 3 & 6 & 6 & 3 & 6 \end{bmatrix} \Longrightarrow \begin{bmatrix} 1 & 2 & 0 & 3 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

2. Solve

$$\left| \begin{array}{cccccc} x_1 & +2x_2 & & +2x_4 & +3x_5 & = 0 \\ & & x_3 & +3x_4 & +2x_5 & = 0 \\ & & x_3 & +4x_4 & -x_5 & = 0 \\ & & & & x_5 & = 0 \end{array} \right|$$

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$$\left[\begin{array}{ccccc|c} 1 & 2 & 0 & 2 & 3 & 0 \\ 0 & 0 & 1 & 3 & 2 & 0 \\ 0 & 0 & 1 & 4 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right] \Longrightarrow \left[\begin{array}{ccccc|c} 1 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

This shows that x_2 is a free variable and thus with $x_2 = t$ we get

$$x_1 = -2t, x_2 = t, x_3 = 0, x_4 = 0, x_5 = 0.$$

3.

(a) Find the orthogonal projection of $\begin{bmatrix} -1 \\ 5 \end{bmatrix}$ onto $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

(b) Find the matrix of the orthogonal projection in part (a).

***************** (a) Here $\vec{x} = \begin{bmatrix} -1 \\ 5 \end{bmatrix}$ and $\vec{u} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Thus

$$Proj_{\vec{u}}\vec{x} = (\vec{x} \cdot \vec{u})\vec{u} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}.$$

(b) The columns of the matrix are $Proj_{\vec{u}}\vec{e}_1 = (\vec{e}_1 \cdot \vec{u})\vec{u} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$. Similarly, $Proj_{\vec{u}}\vec{e}_2 = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$. Hence

$$\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

4. Find the inverse of the following matrix if it exists;

$$\begin{bmatrix} 1 & 0 & 1 \\ 3 & 2 & 0 \\ -1 & 2 & 3 \end{bmatrix}$$

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$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 3 & 2 & 0 & 0 & 1 & 0 \\ -1 & 2 & 3 & 0 & 0 & 1 \end{array}\right] \Longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3/7 & 1/7 & -1/7 \\ 0 & 1 & 0 & -9/14 & 2/7 & 3/14 \\ 0 & 0 & 1 & 4/7 & -1/7 & 1/7 \end{array}\right]$$

5. Give an example of a 2×2 matrix for each of the following linear transformations.

a. Scaling

$$\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

b. Shear

$$\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

c. Rotation

$$\begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

d. Projection

$$\begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$$

c. Reflection

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

6.

a. Demonstrate by example that matrix multiplication is not commutative, -i.e., $AB \neq BA$ in general.

***************** Let, for example, $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Then $AB \neq BA$.

b. For which values of the constant k is the following matrix invertible?

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & k \\ 1 & 9 & k^2 \end{bmatrix}$$

***************** In order for this matrix to be invertible the Gauss-Jordan elimination must reduce A to I_3 in the calculation $[A|I_3] \Longrightarrow [I_3|A^{-1}]$. Now,

$$\begin{bmatrix} 1 & 1 & 1 & | & 1 & 0 & 0 \\ 1 & 2 & k & | & 0 & 1 & 0 \\ 1 & 9 & k^2 & | & 0 & 0 & 1 \end{bmatrix} \Longrightarrow \begin{bmatrix} 1 & 0 & 2-k & | & 2 & 0 & 0 \\ 0 & 1 & k-1 & | & -1 & 1 & 0 \\ 0 & 0 & k^2-8k+7 & | & 7 & -8 & 1 \end{bmatrix}$$

and thus $k^2 - 8k + 7 \neq 0$ for the calculation to proceed. That is, $k \neq 7$ and $k \neq 1$. For other values of k, the matrix is invertible.