Solutions to Selected Problems in HW1

1.2

\# 15 Here we assume that $g\colon [a,b] \to [a,b]$. With $x_0 \in [a,b]$,

$$|x_1 - r| = |g(x_0) - g(r)| \le B|x_0 - r|.$$

Inductively,

$$|x_i - r| \le B|x_{i-1} - r| \le \cdots \le B^i|x_0 - r| \to 0$$

as $i \to \infty$ since $0 < B < 1$.

\# 17 (a) Solving $x - x^3 = x$, we get $x = 0$.

(b) Since $0 < x_0 < 1$, $x_0^3 < x_0$ and

$$0 < x_0 - x_0^3 < x_0 < 1.$$

Similarly we can show that $0 < x_{i+1} < x_i < 1$ and thus

$$x_0 > x_1 > x_2 > \cdots > 0$$

(c) From part (b), the sequence is monotonically decreasing and bounded by below. To show that the sequence converges to 0, we may either show that 0 is the greatest lower bound or use the continuity of $g(x) = x - x^3$. In the latter, if $\lim_{i\to\infty} x_i = L$, $g(L) = g(\lim_{i\to\infty} x_i) = \lim_{i\to\infty} g(x_i) = \lim_{i\to\infty} x_{i+1} = L$ and L is a fixed point. By Part (a), $L = 0$.

1.5

\# 5 With $A = f(a)$, $B = f(b)$ and $C = f(c)$, letting $y = 0$ in (1.35), we get

$$\begin{aligned}
P(0) &= \frac{af(b)f(c)}{(f(a)-f(b))(f(a)-f(c))} + \frac{bf(a)f(c)}{(f(b)-f(a))(f(b)-f(c))} \\
&\quad + \frac{cf(a)f(b)}{(f(c)-f(a))(f(c)-f(b))} \\
&= \frac{a\frac{f(b)-f(c)}{f(a)} + b\frac{f(c)-f(a)}{f(b)} + c\frac{f(a)-f(b)}{f(c)}}{(1-\frac{f(b)}{f(a)})(\frac{f(a)}{f(c)}-1)(1-\frac{f(c)}{f(b)})} \\
&= \frac{as(1-qs) + bqs(r-q) + c(q-1)}{(q-1)(r-1)(s-1)} \\
&= c + \frac{as(1-r) + br(r-q) - c(r^2 - qr - rs + s)}{(q-1)(r-1)(s-1)} \\
&= c + \frac{(c-b)r(r-q) + (c-a)s(1-r)}{(q-1)(r-1)(s-1)}
\end{aligned}$$