Chapter 3- Answers to even problems.

Section 3.1:

14. Note that

$$\begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{bmatrix} \Longrightarrow r.r.e.f.(A) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

and this implies that the first and second columns are linearly independent. Hence $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$ span the image of A.

24. Note that

$$x + 2y + 3z = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

so that the plane $x + 2y + 3z = 0$ is characterized by all vectors $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ that are orthogonal to $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$. Thus the orthogonal projection onto this plane will map $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ to zero vector and thus Kernel=span $\{\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}\}$. Now, the image of this projection consists of those vectors that satisfy $x + 2y + 3z = 0$. It is one equation in three variables and y and z are free variables. Letting $y = s$ and $z = t$, we see that $x = -2s - 3t$. Thus, Image=span$\{\begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}\}$.

30. $A = \begin{bmatrix} 1 & 2 \\ 5 & 10 \end{bmatrix}$.

Section 3.2:

2. W is not a subspace. For example, let $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \in W$. Then $(-1)\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -1 \\ -2 \\ -3 \end{bmatrix} \notin W$.

6. a. Yes b. No (See lecture notes for details.)

8. $(-1)\begin{bmatrix}1\\2\end{bmatrix}+2\begin{bmatrix}2\\3\end{bmatrix}+(-1)\begin{bmatrix}3\\4\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$.

20. Linearly Dependent.

22. $\begin{bmatrix}1&3\\2&6\end{bmatrix} \Longrightarrow r.r.e.f. = \begin{bmatrix}1&3\\0&0\end{bmatrix}$ so that the second column is redundant to the first. Clearly $\begin{bmatrix}3\\6\end{bmatrix}=3\begin{bmatrix}1\\2\end{bmatrix}$.

24. $\begin{bmatrix}1&3&6\\1&2&5\\1&1&4\end{bmatrix} \Longrightarrow r.r.e.f = \begin{bmatrix}1&0&3\\0&1&1\\0&0&0\end{bmatrix}$ show that the third column is three times the first column plus the second column. Namely, $\begin{bmatrix}6\\5\\4\end{bmatrix}=3\begin{bmatrix}1\\1\\1\end{bmatrix}+\begin{bmatrix}3\\2\\1\end{bmatrix}$. Hence

$$3\begin{bmatrix}1\\1\\1\end{bmatrix}+\begin{bmatrix}3\\2\\1\end{bmatrix}-\begin{bmatrix}6\\5\\4\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

This shows that $\begin{bmatrix}3\\1\\-1\end{bmatrix}$ is in the kernel.

26. Arguing similarly to #24, we get $\begin{bmatrix}2\\3\\0\end{bmatrix}=2\begin{bmatrix}1\\0\\0\end{bmatrix}+3\begin{bmatrix}0\\1\\0\end{bmatrix}$ and $\begin{bmatrix}2\\3\\0\end{bmatrix}$ is redundant. $\begin{bmatrix}2\\3\\-1\\0\end{bmatrix}$ is in the kernel.

36. Yes. Since $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m\}$ are linearly dependent, there exist $c_1, c_2, \ldots, c_m$ not all zeroes such that

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_m\vec{v}_m = \vec{0}.$$

Then

$$\begin{aligned}\vec{0} = T(\vec{0}) &= T(c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_m\vec{v}_m)\\ &= c_1T(\vec{v}_1) + c_2T(\vec{v}_2) + \cdots + c_mT(\vec{v}_m).\end{aligned}$$

Since $c_1, c_2, \ldots, c_m$ are not all zeros, we see that $T(\vec{v}_1), T(\vec{v}_2), \ldots, T(\vec{v}_m)$ are linearly dependent.

42. Let $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m\}$ be perpendicular vectors. Consider

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_m\vec{v}_m = \vec{0}.$$

Taking the dot product on both sides with $\vec{v}_i$, $1 \leq i \leq m$,

$$\vec{v}_i \cdot (c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_m\vec{v}_m) = \vec{v}_i \cdot \vec{0} = 0. \qquad (1)$$

As $\vec{v}_j \cdot \vec{v}_i = 0$ if $i \neq j$, and $\vec{v}_i \cdot \vec{v}_i = 1$ as $\vec{v}_i$'s are unit vectors, we see that

$$\vec{v}_i \cdot (c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_m\vec{v}_m) = c_1(\vec{v}_i \cdot \vec{v}_1) + \cdots c_1(\vec{v}_i \cdot \vec{v}_i) + \cdots + c_m(\vec{v}_i \cdot \vec{v}_m) = c_i$$

we obtain from (1) that $c_i = 0$. Hence all the coefficients c_i's are zero, making the vectors linearly independent.

Section 3.3

24. Let

$$A = \begin{bmatrix} 4 & 8 & 1 & 1 & 6 \\ 3 & 6 & 1 & 2 & 5 \\ 2 & 4 & 1 & 9 & 10 \\ 1 & 2 & 3 & 2 & 0 \end{bmatrix}.$$

$$rref(A) = \begin{bmatrix} 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

This shows that the kernel is spanned by $\begin{bmatrix} 2 \\ -1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$. The r.r.e.f. indicates that the first, third, fourth and the fifth columns are linear independent so the corresponding columns of the original A serve as a basis for the image of A.

36. No, as $3 = dim(im(A)) + dim(ker(A))$ by the rank theorem, $dim(im(A)) \neq dim(ker(A))$, so $im(A)$ can not equal $ker(A)$.

38. a. Since $T: R^5 \to R^3$,

$$\begin{array}{rlcc} 5 & = dim(ker(A)) & + & dim(im(A)) \\ & = 5 & + & 0 \\ & = 4 & + & 1 \\ & = 3 & + & 2 \\ & = 2 & + & 3 \end{array}$$

b. Since $T: R^4 \to R^7$,

$$\begin{aligned}
4 &= dim(ker(A)) &+& \quad dim(im(A)) \\
&= 4 &+& \quad 0 \\
&= 3 &+& \quad 1 \\
&= 2 &+& \quad 2 \\
&= 1 &+& \quad 3 \\
&= 0 &+& \quad 4
\end{aligned}$$