Solutions to Quiz 2 - Fall 2009

You must show your work to receive credit.

1. Find the inverse of the transformation

$$y_1 = x_1 + 5x_2$$
$$y_2 = -x_1 + 2x_2$$

**** You may solve the system for x_1 and x_2 in terms of y_1 and y_2, or

$$\begin{bmatrix} 1 & 5 & | & 1 & 0 \\ -1 & 2 & | & 0 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 5 & | & 1 & 0 \\ 0 & 7 & | & 1 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 5 & | & 1 & 0 \\ 0 & 1 & | & \frac{1}{7} & \frac{1}{7} \end{bmatrix}$$

$$\longrightarrow \begin{bmatrix} 1 & 0 & | & \frac{2}{7} & -\frac{5}{7} \\ 0 & 1 & | & \frac{1}{7} & \frac{1}{7} \end{bmatrix}$$

2. Suppose that T is a linear transformation of R^2 to R^2 which rotates a vector counterclockwise by $30°$. Specify the matrix of T.

****

$$\begin{bmatrix} \cos 30° & -\sin 30° \\ \sin 30° & \cos 30° \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}.$$

3. Let L be the line in R^3 that runs parallel to the vector $\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$. Find the orthogonal projection of the vector $\vec{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ onto L.

**** First, we need to find a unit vector $\vec{u}$ in the direction of L. This can be done by

$$\vec{u} = \frac{\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}}{\left\| \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \right\|} = \frac{1}{\sqrt{6}} \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}.$$

Hence

$$Proj_L \vec{x} = (\vec{x} \cdot \vec{u})\vec{u} = \frac{4}{\sqrt{6}} \frac{1}{\sqrt{6}} \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{4}{3} \\ \frac{2}{3} \\ \frac{2}{3} \end{bmatrix}.$$

4. (Bonus - 2pts) Find the matrix of the orthogonal projection in Problem #3.

**** The columns of the matrix are $Proj_L \vec{e}_1$, $Proj_L \vec{e}_2$ and $Proj_L \vec{e}_3$.

$$Proj_L \vec{e}_1 = (\vec{e}_1 \cdot \vec{u})\vec{u} = \left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{bmatrix} \right) \begin{bmatrix} \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{bmatrix} = \begin{bmatrix} \frac{2}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{bmatrix}.$$

Similarly

$$Proj_L \vec{e}_2 = Proj_L \vec{e}_3 = \begin{bmatrix} \frac{1}{3} \\ \frac{1}{6} \\ \frac{1}{6} \end{bmatrix}$$

Hence the matrix of the orthogonal projection is

$$\begin{bmatrix} \frac{2}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{3} & \frac{1}{6} & \frac{1}{6} \end{bmatrix}.$$