MATH 316 - Quiz 3 - Fall 2009 NAME:

You must show your work to receive credit.

1. Find vectors that span the *kernel* of A below. Also, find vectors that span the *image* of A.

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \end{bmatrix}.$$

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Since $r.r.e.f(A) = \begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \end{bmatrix}$, we see that the first and second columns span the *image* of A. Thus

$$im(A) = \text{span}\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix}\}$$

The *kernel* of A can be found by finding all the solutions to $A\vec{x} = \vec{0}$. From the r.r.e.f. of A we see that $x_1 = s + 2t, x_2 = -2s - 3t, x_3 = s$ and $x_4 = t$. Hence

$$im(A) = \text{span}\{\begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \end{bmatrix}\}$$

2. Give an example of a linear transformation whose kernel is the plane $x - y + 3z = 0$.

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Note that $x - y + 3z = [1 \quad -1 \;\; 3] \begin{bmatrix} x \\ y \\ z \end{bmatrix}$. Hence

$$T(\begin{bmatrix} x \\ y \\ z \end{bmatrix}) = [1 \quad -1 \;\; 3] \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

gives a linear transformation whose kernel is $x - y + 3z = 0$.

3. Find a redundant column vector of A below, and write it as a linear combination of the preceding columns.

$$A = \begin{bmatrix} 1 & 3 & 6 \\ 1 & 2 & 5 \\ 1 & 1 & 4 \end{bmatrix}.$$

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$$A \Longrightarrow r.r.e.f.(A) = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

shows that the third column is redundant to the first two as it is 3 times the first column and the second column. Relative to the original matrix A, this means

$$\begin{bmatrix} 6 \\ 5 \\ 4 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}.$$

4. Show that $W = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : x + 2y + 3z = 0 \right\}$ is a subspace of R^3. Exhibit a basis for W.

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First, to find a basis, since the solutions of $x + 2y + 3z = 0$ can be given by $x = -2s - 3t, y = s, z = t$,

$$\{ \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \}$$

is a basis for W.

To show W is a subspace, we must show that W is closed under vector addition and scalar multiplication. To this end, let $\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$ and $\begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}$ be elements of W. Then

$$\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ y_1 + y_2 \\ z_1 + z_2 \end{bmatrix}$$

and the sum is in W since

$$(x_1 + x_2) + 2(y_1 + y_2) + 3(z_1 + z_2) = (x_1 + 2y_1 + 3z_1) + (x_2 + 2y_2 + 3z_2) = 0 + 0 = 0$$

where the penultimate equality was obtained since $x_1 + 2y_1 + 3z_1 = 0$ and $x_2 + 2y_2 + 3z_2 = 0$ as $\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$ and $\begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}$ be elements of W. This show that W is closed under vector addition.

Now for $k \in R$,

$$k \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} kx \\ ky \\ kz \end{bmatrix}$$

and the last vector is in W as $kx + 2ky + 3kz = k(x + 2y + 3z) = k \cdot 0 = 0$. This show that $k \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ is in W and W is closed under the scaler multiplication.