Math 408 - Test 2 - Solution to Selected Problems:

1. Solve the following system by finding the LU factorization of A and followed by forward and backward substitutions. Specify L and U and show your work.

$$\begin{bmatrix} -1 & 1 & 0 & 3 \\ 1 & 0 & 3 & 1 \\ 0 & 1 & -1 & -1 \\ 3 & 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 3 \\ 1 \end{bmatrix}.$$

****

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ -3 & 3 & 2 & 1 \end{bmatrix}, \quad U = \begin{bmatrix} -1 & 1 & 0 & 3 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & -4 & -5 \\ 0 & 0 & 0 & 9 \end{bmatrix}, \quad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1/6 \\ 19/6 \\ -1/6 \\ 1/3 \end{bmatrix}$$

2. Solve the following system in #1 by finding this time the $PA = LU$ factorization and followed by the forward and backward substitutions. Specify P, L and U and show your work.

$$\begin{bmatrix} -1 & 1 & 0 & 3 \\ 1 & 0 & 3 & 1 \\ 0 & 1 & -1 & -1 \\ 3 & 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 3 \\ 1 \end{bmatrix}.$$

****

$$\begin{bmatrix} -1 & 1 & 0 & 3 \\ 1 & 0 & 3 & 1 \\ 0 & 1 & -1 & -1 \\ 3 & 0 & 1 & 2 \end{bmatrix} \underset{P_1}{\longrightarrow} \begin{bmatrix} 3 & 0 & 1 & 2 \\ 1 & 0 & 3 & 1 \\ 0 & 1 & -1 & -1 \\ -1 & 1 & 0 & 3 \end{bmatrix} \begin{matrix} -\frac{1}{3}R_1 + R_2 \to R_2 \\ \longrightarrow \\ \frac{1}{3}R_1 + R_4 \to R_4 \end{matrix}$$

$$\begin{bmatrix} 3 & 0 & 1 & 2 \\ 0 & 0 & 8/3 & 1/3 \\ 0 & 1 & -1 & -1 \\ 0 & 1 & 1/3 & 11/3 \end{bmatrix} \quad \left(\text{note currently} \quad L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1/3 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1/3 & 0 & 0 & 1 \end{bmatrix} \right)$$

$$\underset{P_2}{\longrightarrow} \begin{bmatrix} 3 & 0 & 1 & 2 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 8/3 & 1/3 \\ 0 & 1 & 1/3 & 11/3 \end{bmatrix} \left(\quad L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1/3 & 0 & 1 & 0 \\ -1/3 & 0 & 0 & 1 \end{bmatrix} \right)$$

$$\overset{-R_2+R_4\to R_4}{\longrightarrow}\begin{bmatrix}3&0&1&2\\0&1&-1&-1\\0&0&8/3&1/3\\0&0&4/3&14/3\end{bmatrix}\left(L=\begin{bmatrix}1&0&0&0\\0&1&0&0\\1/3&0&1&0\\-1/3&1&0&1\end{bmatrix}\right)$$

$$\overset{-1/2R_3+R_4\to R_4}{\longrightarrow}\begin{bmatrix}3&0&1&2\\0&1&-1&-1\\0&0&8/3&1/3\\0&0&0&9/2\end{bmatrix}=U$$

In the last step, L is transformed to

$$L=\begin{bmatrix}1&0&0&0\\0&1&0&0\\1/3&0&1&0\\-1/3&1&1/2&1\end{bmatrix}$$

where

$$P_1=\begin{bmatrix}0&0&0&1\\0&1&0&0\\0&0&1&0\\1&0&0&0\end{bmatrix},\quad P_2=\begin{bmatrix}1&0&0&0\\0&0&1&0\\0&1&0&0\\0&0&0&1\end{bmatrix}\quad\text{and}\quad P=P_2P_1=\begin{bmatrix}0&0&0&1\\0&0&1&0\\0&1&0&0\\1&0&0&0\end{bmatrix}$$

5.

a. Find the condition number of $A=\begin{bmatrix}1&2.01\\3&6\end{bmatrix}$. Here use $\|\cdot\|_\infty$ norm.

****

$A^{-1}=\begin{bmatrix}-200&\frac{201}{3}\\100&-\frac{100}{3}\end{bmatrix}$ so that $cond(A)=\|A\|_\infty\|A^{-1}\|_\infty=2403$.

b. Show that $A=\begin{bmatrix}4&1\\1&4\end{bmatrix}$ is positive definite.

****

With $x=\begin{bmatrix}x_1\\x_2\end{bmatrix}\neq 0$,

$$\begin{aligned}x^TAx&=4x_1^2+2x_1x_2+4x_2^2\\&=x_1^2+2x_1x_2+x_2^2+3(x_1^2+x_2^2)\\&=(x_1+x_2)^2+3(x_1^2+x_2^2)>0\end{aligned}$$

6. It is known that if A is symmetric positive definite, then A can be decomposed into an LL^T factorization L^T is the transpose of L, and L is a lower matrix with positive numbers along the diagonal. Find the LL^T factorization of the following 3×3 matrix (here L can be chosen so that the diagonal elements of L are all 1),

$$A = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

******************** Let $L = \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & c & 1 \end{bmatrix}$ and compute LL^T. Compare it to A to find $a = -1, b = 0$ and $c = 1$.

$$L = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$