Math 508 - Test 2 - Name:

1. Solve the following system by finding the $PA = LU$ factorization and followed by the forward and backward substitutions. Specify P, L and U and show your work.

$$\begin{bmatrix} -1 & 1 & 0 & 3 \\ 1 & 0 & 3 & 1 \\ 0 & 1 & -1 & -1 \\ 3 & 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 3 \\ 1 \end{bmatrix}.$$

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$$\begin{bmatrix} -1 & 1 & 0 & 3 \\ 1 & 0 & 3 & 1 \\ 0 & 1 & -1 & -1 \\ 3 & 0 & 1 & 2 \end{bmatrix} \underset{P_1}{\longrightarrow} \begin{bmatrix} 3 & 0 & 1 & 2 \\ 1 & 0 & 3 & 1 \\ 0 & 1 & -1 & -1 \\ -1 & 1 & 0 & 3 \end{bmatrix} \underset{\frac{1}{3}R_1 + R_4 \to R_4}{\overset{-\frac{1}{3}R_1 + R_2 \to R_2}{\longrightarrow}}$$

$$\begin{bmatrix} 3 & 0 & 1 & 2 \\ 0 & 0 & 8/3 & 1/3 \\ 0 & 1 & -1 & -1 \\ 0 & 1 & 1/3 & 11/3 \end{bmatrix} \qquad \left(\text{note currently} \quad L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1/3 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1/3 & 0 & 0 & 1 \end{bmatrix} \right)$$

$$\underset{P_2}{\longrightarrow} \begin{bmatrix} 3 & 0 & 1 & 2 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 8/3 & 1/3 \\ 0 & 1 & 1/3 & 11/3 \end{bmatrix} \overset{-R_2 + R_4 \to R_4}{\longrightarrow} \begin{bmatrix} 3 & 0 & 1 & 2 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 8/3 & 1/3 \\ 0 & 0 & 4/3 & 14/3 \end{bmatrix}$$

$$\left(L \text{ is transformed to} \quad L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1/3 & 0 & 1 & 0 \\ -1/3 & 1 & 0 & 1 \end{bmatrix} \right) \overset{-1/2R_3 + R_4 \to R_4}{\longrightarrow}$$

$$\begin{bmatrix} 3 & 0 & 1 & 2 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 8/3 & 1/3 \\ 0 & 0 & 0 & 9/2 \end{bmatrix} = U$$

In the last step, L is transformed to

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1/3 & 0 & 1 & 0 \\ -1/3 & 1 & 1/2 & 1 \end{bmatrix}$$

where

$$P_1 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad P = P_2 P_1 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

4.

a. Find the condition number of $A = \begin{bmatrix} 1 & 1+\epsilon \\ 1-\epsilon & 1 \end{bmatrix}$. Here use $\| \cdot \|_\infty$ norm.

5.

a. It is known that if A is symmetric positive definite, then A can be decomposed into an LL^T factorization L^T is the transpose of L, and L is a lower matrix with 1's along the diagonal. Find the LL^T factorization of the following 3×3 matrix

$$A = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

b. Describe an algorithm that would accomplish $A = LL^T$ decomposition for a symmetric and positive matrix A.

6. Show that if $Ax = b$ and $A\tilde{x} = \tilde{b}$, then

$$\frac{\|x - \tilde{x}\|}{\|x\|} \le cond(A)\frac{\|b - \tilde{b}\|}{\|b\|}.$$

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(A) $$Ax - A\tilde{x} = b - \tilde{b} \Longrightarrow x - \tilde{x} = A^{-1}(b - \tilde{b}) \Longrightarrow \|x - \tilde{x}\| \le \|A^{-1}\|\|b - \tilde{b}\|$$

Also

(B) $$Ax = b \Longrightarrow \|b\| \le \|A\|\|x\| \Longrightarrow \frac{1}{\|A\|\|x\|} \le \frac{1}{\|b\|} \Longrightarrow \frac{1}{\|x\|} \le \frac{\|A\|}{\|b\|}$$

From (A) and (B),

$$\frac{\|x - \tilde{x}\|}{\|x\|} \le cond(A)\frac{\|b - \tilde{b}\|}{\|b\|}.$$