Math 316 - Sample Final Solutions- Spring 09
Also Review Tests, Quizzes and HW problems.

Final Exam: Tuesday, May 5, 3:45-6:45pm.

1. Find a basis for the image and a basis for the kernel of the matrix

$$A = \begin{bmatrix} 1 & 5 & 4 & 3 & 2 \\ 1 & 6 & 6 & 6 & 6 \\ 1 & 7 & 8 & 10 & 12 \\ 1 & 6 & 6 & 7 & 8 \end{bmatrix}.$$

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Solution: The reduced row echelon form of A, $rref(A)$ is given by

$$rref(A) = \begin{bmatrix} 1 & 0 & -6 & 0 & 6 \\ 0 & 1 & 2 & 0 & -2 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Since the pivot 1 appear in the 1st, 2nd and 4th columns, the corresponding columns of A form a basis for the image of A. Since there is no pivot 1 in the 3rd and 5th column, x_3 and x_5 are free variables, so if we let $x_3 = a$ and $x_5 = b$, then

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = a \begin{bmatrix} 6 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + b \begin{bmatrix} -6 \\ 2 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

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2. Show that $W = \{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} : x_1 - x_2 + 2x_3 + x_4 = 0\}$ is a subspace of R^4. Specify the dimension of W by exhibiting a basis.

Solution: Let $u = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$ and $v = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix}$, then $u + v = \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \\ x_3 + y_3 \\ x_4 + y_4 \end{bmatrix}$ is in W, since

$$(x_1 + y_1) - (x_2 + y_2) + 2(x_3 + y_3) + (x_4 + y_4) =$$
$$(x_1 - x_2 + 2x_3 + x_4) + (y_1 - y_2 + 2y_3 + y_4) = 0$$

where the last equality is justified since u and v are in W. Hence W is closed under the vector addition. Also, for $\lambda \in R$, as $\lambda u = \begin{bmatrix} \lambda x_1 \\ \lambda x_2 \\ \lambda x_3 \\ \lambda x_4 \end{bmatrix}$,

$$\lambda x_1 - \lambda x_2 + 2\lambda x_3 + \lambda x_4 = \lambda(x_1 - x_2 + 2x_3 + x_4) = 0$$

where the last equality is due to the fact that $x_1 - x_2 + 2x_3 + x_4 = 0$. Hence $\lambda u \in W$ and thus W is closed under the scalar multiplication. The above discussion shows that W is a subspace.

$x_1 - x_2 + 2x_3 + x_4 = 0$ is one equation in four unknowns, so there are three free variables, say, x_2, x_3, x_4. Let $x_2 = a, x_3 = b, x_4 = c$, which makes $x_1 = a - 2b - c$ and thus

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = a \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + b \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix} + c \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

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3. Let L be the line through the origin in R^2 that consists of all scalar multiple of $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

   a. Find the projection of $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$ onto L.

   b. Find the reflection of $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$ about L.

   c. Find the matrix of projection onto L.

   d. Find eigenvalues of the matrix in part c.

Solution: (a) With $a = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ and $b = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$, the projection of b onto a, $Proj_a b$, is

$$\frac{a^T b}{a^T a} a = \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}$$

(b) Reflection of b about a, $Ref_a b$, is

$$Ref_a b = 2Proj_a b - b = \begin{bmatrix} -2 \\ -3 \end{bmatrix}$$

(c)

$$\frac{aa^T}{a^T a} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

(d)

$$det \begin{bmatrix} \frac{1}{2}-\lambda & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2}-\lambda \end{bmatrix} = \lambda^2 - \lambda = \lambda(\lambda-1)$$

so 0 and 1 are the eigenvalues of the projection matrix.

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4. Given

$$A = \begin{bmatrix} 1 & 2 & 0 & -2 & 0 \\ 0 & 0 & 5 & 0 & 1 \\ 0 & 3 & 1 & 1 & 0 \end{bmatrix},$$

a. Find the row reduced echelon form of A.

b. If $b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$, then under what conditions on b_1, b_2, b_3 does $Ax = b$ have a solution?

Solution: (a)

$$rref(A) = \begin{bmatrix} 1 & 0 & 0 & -\frac{8}{3} & \frac{2}{15} \\ 0 & 1 & 0 & \frac{1}{3} & -\frac{1}{15} \\ 0 & 0 & 1 & 0 & \frac{1}{5} \end{bmatrix}$$

(b) Part a shows that the column space of A is R^3 (see the pivot 1 appears in the 1st, 2nd and 3rd columns). So any right hand vector b gives a solution to $Ax = b$.

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5.

a. Show that if $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ are linearly independent vectors, then any subset of $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ is linearly independent.

Solution: Let $\{v_{k_1}, \ldots v_{k_m}\}$ be any subset of $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$. To show that it is a linearly independent set, we let $c_{k_1} v_{k_1} + \cdots c_{k_m} v_{k_m} = 0$. Expand this to form a linear combination of $v_1, \ldots, v_n$, $c_1 v_1 + \cdots + c_m v_m = 0$. Here some v_is are $v_{k_1}, \ldots, v_{k_m}$. For example, if $v_{k_1} = v_1$ and $v_{k_2} = v_3$ are taken as elements in the subset, then

$$c_{k_1} v_{k_1} + c_{k_2} v_{k_2} = 0$$

is expanded to

$$c_{k_1} v_{k_1} + c_2 v_2 + c_{k_2} v_3 + \cdots + c_n v_n = 0$$

and this induces $c_{k_1} = c_2 = c_{k_2} = \cdots = c_n = 0$ since $v_1, v_2 \ldots, v_n$ are linearly independent. This shows that $c_{k_1} = c_{k_2} = 0$ and v_{k_1} and v_{k_2} are linearly independent.

b. If $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are linearly independent, then $\vec{v}_1, \vec{v}_1 + \vec{v}_2, \vec{v}_1 + \vec{v}_2 + \vec{v}_3$ are linearly independent vectors.

Solution: Consider

$$c_1\vec{v}_1 + c_2(\vec{v}_1 + \vec{v}_2) + c_3(\vec{v}_1 + \vec{v}_2 + \vec{v}_3) = 0$$

We want to show that $c_1 = c_2 = c_3 = 0$. Now

$$c_1\vec{v}_1 + c_2(\vec{v}_1 + \vec{v}_2) + c_3(\vec{v}_1 + \vec{v}_2 + \vec{v}_3) = 0 \Longrightarrow (c_1 + c_2 + c_3)\vec{v}_1 + (c_2 + c_3)\vec{v}_2 + c_3\vec{v}_3 = 0$$

Since $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are linearly independent, this implies that

$$\begin{aligned} c_1 + c_2 + c_3 &= 0 \\ c_2 + c_3 &= 0 \\ c_3 &= 0 \end{aligned}$$

which gives $c_1 = c_2 = c_3 = 0$.

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6. Given

$$A = \begin{bmatrix} 3 & 3 \\ 3 & -5 \end{bmatrix}$$

Find all eigenvalues of A and corresponding eigenvectors. Find a nonsingular matrix S for which $S^{-1}AS$ is a diagonal matrix Λ. What is Λ?

Solution: $det[A - \lambda I] = (\lambda + 6)(\lambda - 4)$ so $\lambda = -6$ and $\lambda = 4$ are eigenvalues of A. The corresponding eigenvectors are $\begin{bmatrix} -1 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$, respectively. So

$$S = \begin{bmatrix} -1 & 3 \\ 3 & 1 \end{bmatrix}, \quad \Lambda = \begin{bmatrix} -6 & 0 \\ 0 & 4 \end{bmatrix} \quad S^{-1} = \begin{bmatrix} -\frac{1}{10} & \frac{3}{10} \\ \frac{3}{10} & \frac{1}{10} \end{bmatrix}$$

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7.

a. Use the Gram-Schmidt process on the sequence of vectors

$$\left\{ \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

b. Given $A = \begin{bmatrix} -1 & 0 & 1 \\ 2 & 3 & 1 \\ 0 & 3 & 1 \end{bmatrix}$, find orthogonal matrix Q and upper triangular matrix R for which $A = QR$.

Solution: (a) Let

$$a = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ 3 \\ 3 \end{bmatrix}, \quad c = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$q_1 = \frac{a}{\|a\|}$ gives $q_1 = \begin{bmatrix} \frac{-1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \\ 0 \end{bmatrix}$.

$q_2 = \frac{B}{\|B\|}$ where $B = b - (b^T q_1)q_1$ gives $q_2 = \begin{bmatrix} \frac{2}{\sqrt{30}} \\ \frac{1}{\sqrt{30}} \\ \frac{5}{\sqrt{30}} \end{bmatrix}$.

$q_3 = \frac{C}{\|C\|}$ where $C = c - [(c^T q_1)q_1 + (c^T q_2)q_2]$ gives $q_3 = \begin{bmatrix} \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{6}} \end{bmatrix}$.

(b) Recall that $A = OR$ where $Q = [q_1, q_2, q_3]$ and

$$R = \begin{bmatrix} a^T q_1 & b^T q_1 & c^T q_1 \\ 0 & b^T q_2 & c^T q_2 \\ 0 & 0 & c^T q_3 \end{bmatrix} = \begin{bmatrix} \sqrt{5} & \frac{6}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ 0 & \frac{18}{\sqrt{30}} & \frac{8}{\sqrt{30}} \\ 0 & 0 & \sqrt{\frac{2}{3}} \end{bmatrix}$$

Here you should have computed all the components of R in part (a).

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8.

a. If $det(A) = 2$ and $det(2A) = 64$, then what is the size of A? Explain.
b. If $B, C \in R^{3\times 3}$ and $det(B) = 4$ and $det(C) = -2$, what is $det(3BC)$? Explain.
c. If $A = P^{-1}BP$ and $det(A) = 5$, then what is $det(B)$? Explain.

Solution: (a) If A is $n \times n$, then $det(2A) = 2^n det(A) = 2^{n+1}$. Setting the last expression equal 64, we get $n = 5$.

(b) Since B and C are 3×3, $det(3BC) = 3^3 det(B)det(C) = -216$.

(c) Since $det(A) = det(P^{-1}BP) = det(P^{-1})det(B)det(P) = det(B)$, $det(B) = 5$.

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9. Consider the matrix $B = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 \end{bmatrix}$.

   a. Find the eigenvalues and the corresponding eigenvectors.

   b. Is B diagonalizable? If so, find S and D such that $D = S^{-1}BS$.

Solution:

$$det[B - \lambda I] = det \begin{bmatrix} -\lambda & 0 & -1 & 0 \\ 0 & 1-\lambda & 0 & 0 \\ 0 & 0 & 1-\lambda & 0 \\ 1 & 0 & 1 & 1-\lambda \end{bmatrix} = -\lambda(1-\lambda)^3$$

Here the best way to evaluate the determinant above is the cofactor expansion along the the 4th column as it contains three 0s. Hence the eigenvalues of B are 0 and 1 with respective algebraic multiplicities of 1 and 3.

$\begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}$ is an eigenvector for $\lambda = 0$ and

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

are eigenvectors for $\lambda = 1$. Since there are three linearly independent eigenvectors for $\lambda = 3$, its geometric multiplicity is 3 which agrees with its algebraic multiplicity. Hence B is diagonalizable.

$$S = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

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10. Consider the following set of vectors in R^3.

   a $S_1 = \{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 4 \end{bmatrix} \}$

   b $S_2 = \{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \}$

c $S_3 = \{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \}$

d $S_4 = \{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \}$

Which of these sets are linearly independent? Which of them span R^3? Which of them form bases for R^3?

Solution:(a) For S_1, form the following matrix and do row reduction.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 3 \\ 1 & 1 & 4 \end{bmatrix} \Longrightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Hence three columns are linearly independent. Also

$$\begin{bmatrix} 1 & 0 & 0 & | & a \\ 0 & 1 & 3 & | & b \\ 1 & 1 & 4 & | & c \end{bmatrix} \Longrightarrow \begin{bmatrix} 1 & 0 & 0 & | & a \\ 0 & 1 & 0 & | & 2b-3c+3a \\ 0 & 0 & 1 & | & c-a-b \end{bmatrix}$$

implies that any vector $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ in R^3 can be written as a linear combination of three vectors in S_1. Hence S_1 span R^3. S_1, being linearly independent and span R^3, is a basis.

Similarly one can show

(b) S_2 is neither linearly independent nor span R^3, hence not a basis.

(c) S_3 is not linearly independent, but it spans R^3, hence not a basis.

(d) S_4 is linearly independent but does not span R^3, hence not a basis.

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11.

a. Find the determinant of

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 2 & 9 & 1 & 0 \\ 0 & 9 & 0 & 0 \\ 1 & 9 & 9 & 5 \end{bmatrix}$$

b. Find the determinant of

$$A = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & 2 & 2 & \cdots & 2 \\ 1 & 2 & 3 & \cdots & 3 \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & 2 & 3 & \cdots & n \end{bmatrix}$$

Solution: (a) You may use row operation or cofactor expansion to compute the determinant. For cofactor expansion, expand along the 4th column since it contains all zero except 5 in the $(4,4)$ position.

$$detA = det\begin{bmatrix}1 & 1 & 0 & 0\\ 2 & 9 & 1 & 0\\ 0 & 9 & 0 & 0\\ 1 & 9 & 9 & 5\end{bmatrix} = 5det\begin{bmatrix}1 & 1 & 0\\ 2 & 9 & 1\\ 0 & 9 & 0\end{bmatrix} = 5\cdot(-1)\cdot 9det\begin{bmatrix}1 & 0\\ 2 & 1\end{bmatrix} = -45$$

where the third equality is obtained by cofactor expansion along the 3rd row.

(b) Let's take the case for $n = 4$. Using the row operations,

$$detA = det\begin{bmatrix}1 & 1 & 1 & 1\\ 1 & 2 & 2 & 2\\ 1 & 2 & 3 & 3\\ 1 & 2 & 3 & 4\end{bmatrix} = det\begin{bmatrix}1 & 1 & 1 & 1\\ 0 & 1 & 1 & 1\\ 0 & 1 & 2 & 2\\ 0 & 1 & 2 & 3\end{bmatrix} = det\begin{bmatrix}1 & 1 & 1 & 1\\ 0 & 1 & 1 & 1\\ 0 & 0 & 1 & 1\\ 0 & 0 & 1 & 2\end{bmatrix}$$

$$= det\begin{bmatrix}1 & 1 & 1 & 1\\ 0 & 1 & 1 & 1\\ 0 & 0 & 1 & 1\\ 0 & 0 & 0 & 1\end{bmatrix} = 1$$

Generalize this to any n, we see that $detA = 1$.

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12. Solve $\frac{du(t)}{dt} = \begin{bmatrix}\frac{1}{2} & \frac{1}{2}\\ \frac{1}{2} & \frac{1}{2}\end{bmatrix} u(t)$, $u(0) = \begin{bmatrix}1\\ -1\end{bmatrix}$.

Solution: From #6 of Test 2, $\lambda = 0$ and $\lambda = 1$ are the eigenvalues with corresponding eigenvectors of $\begin{bmatrix}1\\ -1\end{bmatrix}$ and $\begin{bmatrix}1\\ 1\end{bmatrix}$ respectively. Let $S = \begin{bmatrix}1 & 1\\ -1 & 1\end{bmatrix}$, so that

$$u(t) = Se^{\Lambda t}S^{-1}u(0) = \begin{bmatrix}1 & 1\\ -1 & 1\end{bmatrix}\begin{bmatrix}e^{0t} & 0\\ 0 & e^{1t}\end{bmatrix}\begin{bmatrix}\frac{1}{2} & -\frac{1}{2}\\ \frac{1}{2} & \frac{1}{2}\end{bmatrix}\begin{bmatrix}1\\ -1\end{bmatrix}$$

$$= \begin{bmatrix}1 & 1\\ -1 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & e^{t}\end{bmatrix}\begin{bmatrix}\frac{1}{2} & -\frac{1}{2}\\ \frac{1}{2} & \frac{1}{2}\end{bmatrix}\begin{bmatrix}1\\ -1\end{bmatrix} = \begin{bmatrix}1\\ -1\end{bmatrix}$$

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13. For $A = \begin{bmatrix}\frac{1}{2} & \frac{3}{4}\\ \frac{1}{2} & \frac{1}{4}\end{bmatrix}$,

a. Find A^k.

b. Find $\lim_{k\to\infty} A^k$.

Solution: (a) Routine calculation shows that $\lambda = -\frac{1}{4}$ and $\lambda = 1$ are the eigenvalues of A and their corresponding eigenvectors are $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ for $-\frac{1}{4}$ and $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$ for 1. So if we let $S = \begin{bmatrix} 1 & 3 \\ -1 & 2 \end{bmatrix}$ and $\Lambda = \begin{bmatrix} -\frac{1}{4} & 0 \\ 0 & 1 \end{bmatrix}$, then $A = S\Lambda S^{-1}$ and

$$A^k = S\lambda^k S^{-1} = \begin{bmatrix} 1 & 3 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} (-\frac{1}{4})^k & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{2}{5} & -\frac{3}{5} \\ \frac{1}{5} & \frac{1}{5} \end{bmatrix}$$

(b) Since $\lim_{k\to\infty}(-\frac{1}{4})^k = 0$, from part (a),

$$\lim_{k\to\infty} A^k = \begin{bmatrix} 1 & 3 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{2}{5} & -\frac{3}{5} \\ \frac{1}{5} & \frac{1}{5} \end{bmatrix} = \begin{bmatrix} \frac{3}{5} & \frac{3}{5} \\ \frac{2}{5} & \frac{2}{5} \end{bmatrix}$$

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14. Exercise #14, p. 264.

Solution: (a) Relative to a vector $\begin{bmatrix} America(k) \\ Asia(k) \\ Europe(k) \end{bmatrix}$, the information provided gives rise to the matrix equation describing this discrete dynamical system:

$$\begin{bmatrix} America(k+1) \\ Asia(k+1) \\ Europe(k+1) \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{2} & 0 \\ \frac{1}{4} & 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} America(k) \\ Asia(k) \\ Europe(k) \end{bmatrix}$$

(b) The eigenvalues of this 3×3 matrix are $0, \frac{1}{2}, 1$. Corresponding eigenvectors are

$$\lambda_0 = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}, \lambda_1 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \lambda_{\frac{1}{2}} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

(c) Since $\begin{bmatrix} America(0) \\ Asia(0) \\ Europe(0) \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}$,

$$\begin{aligned} \begin{bmatrix} America(k) \\ Asia(k) \\ Europe(k) \end{bmatrix} &= A^k \begin{bmatrix} America(0) \\ Asia(0) \\ Europe(0) \end{bmatrix} \\ &= S\Lambda^k S^{-1} \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix} \end{aligned}$$

where

$$S = \begin{bmatrix} -2 & 2 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & -1 \end{bmatrix}, \quad \Lambda = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}, \quad S^{-1} = \begin{bmatrix} -\frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ 0 & \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$$

Hence

$$\lim_{k\to\infty} \Lambda^k = \lim_{k\to\infty} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1^k & 0 \\ 0 & 0 & (\frac{1}{2})^k \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and

$$\lim_{k\to\infty} \begin{bmatrix} America(k) \\ Asia(k) \\ Europe(k) \end{bmatrix} = \begin{bmatrix} -2 & 2 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} -\frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ 0 & \frac{1}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$$

(d) From part (c),

$$\begin{bmatrix} America(k) \\ Asia(k) \\ Europe(k) \end{bmatrix} = S\Lambda^k S^{-1} \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}$$

which reduces to

$$\begin{bmatrix} -2 & 2 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1^k & 0 \\ 0 & 0 & (\frac{1}{2})^k \end{bmatrix} \begin{bmatrix} -\frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ 0 & \frac{1}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ 1 - (\frac{1}{2})^{k+1} \\ 1 + (\frac{1}{2})^{k+1} \end{bmatrix}$$