Notes on Section 2.4

In this section, we learn four important subspaces in relation to an $m \times n$ matrix A. They are

1. $C(A)$, the column space of A.
2. $N(A)$, the null space of A.
3. $R(A)$, the row space of A, this is equal to $C(A^T)$.
4. $LN(A)$, the left null space of A, this is equal to $N(A^T)$.

Example: (exercise 3) Consider

$$A = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{bmatrix} \quad \text{and} \quad U = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The RREF of A is

$$RREF(A) = \begin{bmatrix} 1 & 0 & -2 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Since the $RREF(A)$ contains the pivot 1 in the first and the second columns, these columns in the matrix A can be used for a basis of $C(A)$. (Note: you may not use the first two columns in $RREF(A)$ to form a basis for $C(A)$, since they generate a difference space.) Thus, the dimension r of $C(A)$ is 2 and a basis for $C(A)$ is

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} \right\}.$$

The dimension of $C(A)$ which is equal to the dimension of $R(A)$ is called the **rank** of A.

Now, for $N(A)$, we need to characterize a solution to homogeneous equation $Ax = 0$. From the $RREF(A)$, we see that $x = [u, v, w, z]^T$ must satisfy, with $w = a$ and $z = b$,

$$\begin{bmatrix} u \\ v \\ w \\ z \end{bmatrix} = \begin{bmatrix} 2a + b \\ -a \\ a \\ b \end{bmatrix} = a \begin{bmatrix} 2 \\ -1 \\ 1 \\ 0 \end{bmatrix} + b \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

Hence

$$\left\{ \begin{bmatrix} 2 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

is a basis for $N(A)$ and the dimension of $N(A)$ is 2. The dimension of $N(A)$ is also called as the **nullity** of A.

Notice that in the example above, A has four columns and $4 = rank(A) + nullity(A)$. This is true in general and it is called **the rank-nullity theorem.** This says that if A is an $m \times n$ matrix, then $n = rank(A) + nullity(A)$.

To describe the row space of A, $R(A)$, once again note that the pivot 1 is located in the first and in the second row of $RREF(A)$. This tells us that the first and the second rows of A are linearly independent. To describe a basis for $R(A)$, we may select the first and the second rows of A or the first and the second rows of $RREF(A)$. This is different from the selection process for a basis for $C(A)$ as described above. The reason underpinning this fact is that $RREF(A)$ is obtained from A by successive applications of elementary row operations, each of which does not change the linear independency of the rows of A. Hence a basis for $R(A)$ is

$$\{(1\ 2\ 0\ 1), \quad (0\ 1\ 1\ 0)\}$$

and the dimension of $R(A)$ is two which is the rank of A. Now

$$A^T = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 2 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \Longrightarrow RREF(A^T) = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Examining the pivot 1 in $RREF(A)$, we see that $C(A^T)$ has a basis

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \right\}$$

Hence $R(A) = C(A^T)$.

The left null space of A, $LN(A)$, is the null space of A^T, $N(A^T)$. It is easy to see by examining $RREF(A^T)$ that a basis for $N(A^T)$ is given by

$$\left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

$3 = rank(A^T) + nullity(A^T)$.