

Solutions to Selected Even Numbered Exercises in Chapter 2

Section 2.1.

2.

- (a) Yes
- (b) No
- (c) No
- (d) Yes
- (e) Yes

8.

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \end{bmatrix} \implies RREF(A) = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \end{bmatrix}$$

So z is free and a basis of $N(A)$ is

$$\left\{ \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} \right\}.$$

Hence

- (a) No
- (b) Yes
- (c) No
- (d) Yes
- (e) Yes
- (f) No

22.

(a) Note that the system is solvable provided that the right side vector $\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ is in

the column space of A . Now

$$A = \begin{bmatrix} 1 & 4 & 2 \\ 2 & 8 & 4 \\ -1 & -4 & -2 \end{bmatrix} \implies RREF(A) = \begin{bmatrix} 1 & 4 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Since the only pivot 1 appears in the first column, the column space has a basis

(chosen from the first column of A), $\left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \right\}$. Thus the system is solvable if

$$\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = c \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \text{ for some } c \in R.$$

(b) Here,

$$A = \begin{bmatrix} 1 & 4 \\ 2 & 9 \\ -1 & -4 \end{bmatrix} \implies RREF(A) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Since the pivot 1 appears in the first and the second columns of A , $C(A)$ has a basis $\left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 4 \\ 9 \\ -4 \end{bmatrix} \right\}$ and the system is solvable provided that $\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ is in this column space $C(A)$. More precisely,

$$\left[\begin{array}{cc|c} 1 & 4 & b_1 \\ 2 & 9 & b_2 \\ -1 & -4 & b_3 \end{array} \right] \implies \left[\begin{array}{cc|c} 1 & 4 & b_1 \\ 2 & 9 & b_2 \\ 0 & 0 & b_1 + b_3 \end{array} \right]$$

so the system is solvable provided $b_1 + b_3 = 0$.

Section 2.2.

2.

$$A = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{bmatrix} \implies RREF(A) = \begin{bmatrix} 1 & 0 & -2 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Hence x_3 and x_4 are free variables and the rank of A is 2. A general solution is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2x_3 - x_4 \\ -x_3 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} 2 \\ -1 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

To get a particular solution, choose any number for x_3 and x_4 . For example, with

$$x_3 = 1 \text{ and } x_4 = 1, \text{ we get } \begin{bmatrix} 1 \\ -1 \\ 1 \\ 1 \end{bmatrix}.$$

An analysis for B is similar.

6.

$$\left[\begin{array}{ccc|c} 1 & 0 & b_1 \\ 0 & 1 & b_2 \\ 2 & 3 & b_3 \end{array} \right] \implies \left[\begin{array}{ccc|c} 1 & 0 & b_1 \\ 0 & 1 & b_2 \\ 0 & 0 & b_3 - 2b_1 - 3b_2 \end{array} \right]$$

Observing the last row, we conclude that a solution exists provided $b_3 - 2b_1 - 3b_2 = 0$.

$x_1 = b_1$ and $x_2 = b_2$ are particular solution and the rank of the matrix is 2.

8.

$$\left[\begin{array}{cccc|c} 1 & 2 & 0 & 3 & b_1 \\ 2 & 4 & 0 & 7 & b_2 \end{array} \right] \implies \left[\begin{array}{cccc|c} 1 & 2 & 0 & 0 & 7b_1 - 3b_2 \\ 0 & 0 & 0 & 1 & b_2 - 2b_1 \end{array} \right]$$

Hence, a solution exists without any conditions on b_1 and b_2 . Examining the last

expression, we see that a particular solution can be given as $\begin{bmatrix} 7b_1 - 3b_2 - 2x_2 \\ x_2 \\ x_3 \\ b_2 - 2b_1 \end{bmatrix}$ for

any x_2 and x_3 . If $x_2 = x_3 = 0$, then a particular solution $\begin{bmatrix} 7b_1 - 3b_2 \\ 0 \\ 0 \\ b_2 - 2b_1 \end{bmatrix}$ is obtained.

To find the null space of A , -i.e., solutions when $b_1 = b_2 = 0$,

$$\begin{bmatrix} 7b_1 - 3b_2 - 2x_2 \\ x_2 \\ x_3 \\ b_2 - 2b_1 \end{bmatrix} = \begin{bmatrix} -2x_2 \\ x_2 \\ x_3 \\ 0 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

and thus

$$\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

is a basis for the null space $N(A)$ of A .

34

$$\left[\begin{array}{ccc|c} 1 & 2 & -2 & b_1 \\ 2 & 5 & -4 & b_2 \\ 4 & 9 & -8 & b_3 \end{array} \right] \implies \left[\begin{array}{ccc|c} 1 & 0 & -2 & 5b_1 - 2b_2 \\ 0 & 1 & 0 & b_2 - 2b_1 \\ 0 & 0 & 0 & 2b_1 + b_2 - b_3 \end{array} \right]$$

Hence the system is solvable (= consistent) if $2b_1 + b_2 - b_3 = 0$. In this case, the last expression becomes

$$\left[\begin{array}{ccc|c} 1 & 0 & -2 & 5b_1 - 2b_2 \\ 0 & 1 & 0 & b_2 - 2b_1 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

and since the first and the second columns contain the pivot 1, we see that x_3 is free variable and a general solution can be given as

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5b_1 - 2b_2 + 2x_3 \\ b_2 - 2b_1 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5b_1 - 2b_2 \\ b_2 - 2b_1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$$

where $\begin{bmatrix} 5b_1 - 2b_2 \\ b_2 - 2b_1 \\ 0 \end{bmatrix}$ is a particular solution and $\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$ is a homogeneous solution.

Section 2.3:

2.

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 1 & 0 \\ 0 & -1 & 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 0 & -1 & -1 \end{bmatrix} \implies RREF = \begin{bmatrix} 1 & 0 & 0 & -1 & -1 & 0 \\ 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Since the pivot 1 appears in the 1st, 2nd and 3rd columns, v_1, v_2, v_3 are linearly independent and all the other v 's are linearly dependent to them. So the dimension of the space spanned by $\{v_1, v_2, \dots, v_6\}$ is 3.

6.

$$U = \begin{bmatrix} 2 & 3 & 4 & 1 \\ 0 & 6 & 7 & 0 \\ 0 & 0 & 0 & 9 \\ 0 & 0 & 0 & 0 \end{bmatrix} \implies RREF(U) = \begin{bmatrix} 1 & 0 & 1/4 & 0 \\ 0 & 1 & 7/6 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Since there are three pivot 1's, the dimension of the column space of U is 3. For a set of three linearly independent columns, we can select $\{1, 2, 4\}$, $\{1, 3, 4\}$ or $\{2, 3, 4\}$. The columns $\{1, 2, 3\}$ are linearly dependent.

Note that the second matrix A reduces to U , so the same results.

8. Consider $c_1v_1 + c_2v_2 + c_3v_3 = 0$. To show that v_1, v_2 and v_3 are linearly independent, we must show that $c_1 = c_2 = c_3 = 0$. Since $v_1 = w_2 + w_3$, $v_2 = w_1 + w_3$ and $v_3 = w_1 + w_2$,

$$c_1v_1 + c_2v_2 + c_3v_3 = 0 \implies c_1(w_2 + w_3) + c_2(w_1 + w_3) + c_3(w_1 + w_2) = 0$$

The last equation simplifies to

$$(c_2 + c_3)w_1 + (c_1 + c_3)w_2 + (c_1 + c_2)w_3 = 0.$$

Since w_1, w_2 and w_3 are linearly independent by assumption, the last equation forces $c_2 + c_3 = 0$, $c_1 + c_3 = 0$ and $c_1 + c_2 = 0$, or

$$\begin{bmatrix} 0 & 1 & 1 & | & 0 \\ 1 & 0 & 1 & | & 0 \\ 1 & 1 & 0 & | & 0 \end{bmatrix} \implies \begin{bmatrix} 1 & 0 & 0 & | & 0 \\ 0 & 1 & 0 & | & 0 \\ 0 & 0 & 1 & | & 0 \end{bmatrix}.$$

Thus the only solution is the zero solution $c_1 = c_2 = c_3 = 0$ and this shows that v_1, v_2, v_3 are linearly independent.

18. (a) $Ax = b$ has no solution, so b is not in the subspace spanned by w_1, w_2, w_3 .

(b) w 's (with or without w_4) span R^3 , producing every b .

20.

(a) $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\}$

(b) For this subspace of R^4 , each element $\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$ must satisfy $a + b + c + d = 0$. This is one equation in four unknowns, a, b, c, d , so three free variables, say b, c, d . Thus

$$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} -b - c - d \\ b \\ c \\ d \end{bmatrix} = b \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + c \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + d \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

and a basis is given by

$$\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

(c) If $\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$ is perpendicular to $\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}$, then their inner products must be zero, -i.e.,

$$\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} = x + y = 0, \quad \text{and} \quad \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} = x + z + w = 0.$$

Solving equations $x + y = 0$ and $x + z + w = 0$ from

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \end{array} \right] \implies \left[\begin{array}{cccc|c} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & -1 & -1 & 0 \end{array} \right]$$

with $z = c$ and $w = d$, we get

$$\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} -c - d \\ c + d \\ c \\ d \end{bmatrix} = c \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix} + d \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

so a basis for this subspace is

$$\left\{ \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

(d) $U = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix}$ is RREF. Since the pivot 1 is on the 1st and on the 2nd columns, the column space of U , $C(U)$, span R^2 . A basis is

$$\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}.$$

The null space, $N(U)$, contains the solutions x of $Ux = 0$. Hence

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -x_3 - x_5 \\ -x_4 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Hence a basis is given by

$$\left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

24. From calculus, we know that $x - 2y + 3z = 0$ is a plane in R^3 . If $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ is a vector on

this plane, then

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2y - 3z \\ y \\ z \end{bmatrix} = y \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$$

so a basis for this plane is

$$\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \right\}$$

On the xy -plane, $z = 0$. So $x - 2y + 3z = 0 \implies x - 2y = 0$. So a vector $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ which lies on the intersection must satisfy

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2y \\ y \\ 0 \end{bmatrix} = y \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

and

$$\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \right\}$$

is a basis for the line of intersection.

A vector $\begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$ is perpendicular to the plane.

Section 2.4: For the problems in this section, refer to the handout which was distributed on Wed. Feb. 18th.

2.

- $C(A)$, the column space of A .
- $N(A)$, the null space of A .
- $R(A)$, the row space of A , this is equal to $C(A^T)$.
- $LN(A)$, the left null space of A , this is equal to $N(A^T)$.

$$A = \begin{bmatrix} 0 & 1 & 4 & 0 \\ 0 & 2 & 8 & 0 \end{bmatrix} \quad \text{and} \quad U = \begin{bmatrix} 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The RREF of A is

$$RREF(A) = \begin{bmatrix} 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Since the $RREF(A)$ contains the pivot 1 in the second column, the second column in the matrix A can be used for a basis of $C(A)$. Thus, the dimension r of $C(A)$ is 1 and a basis for $C(A)$ is

$$\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}.$$

Hence the rank of A is 1.

Now, for $N(A)$, we need to characterize a solution to homogeneous equation $Ax = 0$. From the $RREF(A)$, we see that $x = [u, v, w, z]^T$ must satisfy $v + 4w = 0$. With $u = a$, $w = b$ and $z = c$,

$$\begin{bmatrix} u \\ v \\ w \\ z \end{bmatrix} = \begin{bmatrix} a \\ -4b \\ b \\ c \end{bmatrix} = a \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + b \begin{bmatrix} 0 \\ -4 \\ 1 \\ 0 \end{bmatrix} + c \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

Hence

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -4 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

is a basis for $N(A)$ and the dimension of $N(A)$ is 3, or the nullity of A is 3.

Notice that in the example above, A has four columns and $4 = \text{rank}(A) + \text{nullity}(A)$.

To describe the row space of A , $R(A)$, once again note that the pivot 1 is located in the first row of $RREF(A)$. This tells us that the first row of A alone spans $R(A)$. To describe

a basis for $R(A)$, we may select the first row of A or the first and row of $RREF(A)$, which in this case is the same. Hence a basis for $R(A)$ is

$$\{(0 \ 1 \ 4 \ 0)\}.$$

Now

$$A^T = \begin{bmatrix} 0 & 0 \\ 1 & 2 \\ 4 & 8 \\ 0 & 0 \end{bmatrix} \implies RREF(A^T) = \begin{bmatrix} 0 & 0 \\ 1 & 2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Examining the pivot 1 in $RREF(A^T)$, we see that $C(A^T)$ has a basis

$$\left\{ \begin{bmatrix} 0 \\ 1 \\ 4 \\ 0 \end{bmatrix} \right\}$$

Hence $R(A) = C(A^T)$.

The left null space of A , $LN(A)$, is the null space of A^T , $N(A^T)$. It is easy to see by examining $RREF(A^T)$ that a basis for $N(A^T)$ is given by

$$\left\{ \begin{bmatrix} -2 \\ 1 \end{bmatrix} \right\}.$$

Now for

$$U = \begin{bmatrix} 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$C(U)$ has a basis

$$\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$$

Also, $N(U) = N(A)$ since $U = RREF(A)$. Similarly, $R(U) = R(A)$ and $LN(U) = N(U^T)$ has a basis

$$\left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}.$$

Section 2.6

- # 2. Here consider the reflection to be about the x -axis and the rotation to be by the angle π radian. Then the five reflections amounts to one reflection and eight rotations that follow would put the plane to the same place. So it amounts to a reflection.
- # 4. If z is half way between x and y , then $z = \frac{x+y}{2}$. This is because $z = x + \frac{y-x}{2} = \frac{x+y}{2}$. Then $A(z) = A(\frac{x+y}{2}) = \frac{A(x)+A(y)}{2}$ where the last equality is obtained using the definition of the linear transformation. Hence $A(z)$ is halfway between $A(x)$ and $A(y)$.